ON A FAMILY OF HOPF ALGEBRAS OF DIMENSION 72

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Abstract. We investigate a family of Hopf algebras of dimension 72 whose coradical is isomorphic to the algebra of functions on $\mathbb{S}_3$. We determine the lattice of submodules of the so-called Verma modules and as a consequence we classify all simple modules. We show that these Hopf algebras are unimodular (as well as their duals) but not quasitriangular; also, they are cocycle deformations of each other.

Introduction

The study of finite dimensional Hopf algebras over an algebraically closed field $\Bbbk$ of characteristic 0 is split into two different classes: the class of semisimple Hopf algebras and the rest. The Lifting Method from [AS] is designed to deal with non-semisimple Hopf algebras whose coradical is a Hopf subalgebra[1]. Pointed Hopf algebras, that is Hopf algebras whose coradical is a group algebra, were intensively studied by this Method. It is natural to consider next the class of Hopf algebras whose coradical is the algebra $\Bbbk^G$ of functions on a non-abelian group G. This class seems to be interesting at least by the following reasons:

• The categories of Yetter-Drinfeld modules over the group algebra $\Bbbk G$ and $\Bbbk^G$, G a finite group, are equivalent. Thence, a lot sensible information needed for the Lifting Method (description of Yetter-Drinfeld modules, determination of finite dimensional Nichols algebras) can be translated from the pointed case to this case –or vice versa.

• The representation theory of Hopf algebras whose coradical is the algebra of functions on a non-abelian group looks easier that the the representation theory of pointed Hopf algebras with non-abelian group, because the representation theory of $\Bbbk^G$ is easier than that of G. Indeed, $\Bbbk^G$ is a semisimple abelian algebra and we may try to imitate the rich methods in representation theory of Lie algebras, with $\Bbbk^G$ playing the role of the Cartan subalgebra. We believe that the representation theory of Hopf algebras with coradical $\Bbbk^G$ might be helpful to study Nichols algebras and deformations.

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[1]An adaptation to general non-semisimple Hopf algebras was recently proposed in [AC].

We have started the consideration of this class in [AV], where finite dimensional Hopf algebras whose coradical is $\Bbbk^{\mathbb{S}_3}$ were classified and, in particular, a new family of Hopf algebras of dimension 72 was defined. The purpose of the present paper is to study these Hopf algebras. We first discuss in Section 1 some general ideas about modules induced from simple $\Bbbk^G$- modules, that we call Verma modules. We introduce in Section 2 a new family of Hopf algebras, as a generalization of the construction in [AV], attached to the class of transpositions in $\mathbb{S}_n$ and depending on a parameter $\mathbf{a}$.

Our main contributions are in Section 3: we determine the lattice of submodules of the various Verma modules and as a consequence we classify all simple modules over the Hopf algebras of dimension 72 introduced in [AV]. Some further information on these Hopf algebras is given in Section 4 and Section 5.

We assume that the reader has some familiarity with Yetter-Drinfeld modules and Nichols algebras $\mathcal{B}(V)$; we refer to [AS] for these matters.

Conventions.

If V is a vector space, $T(V)$ is the tensor algebra of V. If S is a subset of V, then we denote by $\langle S\rangle$ the vector subspace generated by S. If A is an algebra and S is a subset of A, then we denote by (S) the two-sided ideal generated by S and by $\Bbbk\langle S\rangle$ the subalgebra generated by S. If H is a Hopf algebra, then Δ, ϵ, $\mathcal{S}$ denote respectively the comultiplication, the counit and the antipode. We denote by $\widehat{R}$ the set of isomorphism classes of a simple R-modules, R an algebra; we identify a class in $\widehat{R}$ with a representative without further notice. If S, T and M are R-modules, we say that M *is an extension of* T *by* S when M fits into an exact sequence $0 \to S \to M \to T \to 0$.

1. Preliminaries

1.1. **The induced representation.**

We collect well-known facts about the induced representation. Let B be a subalgebra of an algebra A and let V be a left B-module. The induced module is $\operatorname{Ind}_B^A V = A \otimes_B V$. The induction has the following properties:

- Universal property: if W is an A-module and $\varphi : V \to W$ is morphism of B-modules, then it extends to a morphism of A-modules $\overline{\varphi} : \operatorname{Ind}_B^A V \to W$. Hence, there is a natural isomorphism (called Frobenius reciprocity): $\operatorname{Hom}_B(V, \operatorname{Res}_B^A W) \simeq \operatorname{Hom}_A(\operatorname{Ind}_B^A V, W)$. In categorical terms, *induction is left-adjoint to restriction*.
- Any finite dimensional simple A-module is a quotient of the induced module of a simple B-module.

Indeed, let S be a finite dimensional simple A-module and let T be a simple B-submodule of S. Then the induced morphism $\operatorname{Ind}_B^A T \to S$ is surjective.

- If B is semisimple, then any induced module is projective.

The induction functor, being left adjoint to the restriction one, preserves projectives, and any module over a semisimple algebra is projective.

- If A is a free right B-module, say $A \simeq B^{(I)}$, then $\operatorname{Ind}_B^A V = B^{(I)} \otimes_B V = V^{(I)}$ as B-modules, and a fortiori as vector spaces.

We summarize these basic properties in the setting of finite dimensional Hopf algebras, where freeness over Hopf subalgebras is known [NZ]. Also, finite dimensional Hopf algebras are Frobenius, so that injective modules are projective and vice versa.

Proposition 1. *Let A be a finite dimensional Hopf algebra and let B be a semisimple Hopf subalgebra.*

- *If $T \in \widehat{B}$, then* $\dim \operatorname{Ind}_B^A T = \dfrac{\dim T \dim A}{\dim B}$.
- *Any finite dimensional simple A-module is a quotient of the induced module of a simple B-module.*
- *The induced module of a finite dimensional B-module is injective and projective.* □

1.2. **Representation theory of Hopf algebras with coradical a dual group algebra.**

An optimal situation to apply the Proposition 1 is when the coradical of the finite dimensional Hopf algebra A is a Hopf subalgebra; in this case $B =$ coradical of A is the best choice. It is tempting to say that the induced module of a simple B-module is a *Verma module* of A.

Assume now the coradical B of the finite dimensional Hopf algebra A is the algebra of functions $\Bbbk^G$ on a finite group G. In this case, we have:

• Any simple B-module has dimension 1 and $\widehat{B} \simeq G$; for $g \in G$, the simple module $\Bbbk_g$ has the action $f \cdot 1 = f(g)1$, $f \in \Bbbk^G$. Thus any simple A-module is a quotient of a Verma module $M_g := \operatorname{Ind}_{\Bbbk^G} \Bbbk_g$, for some $g \in G$.

• The ideal $A\delta_g$ is isomorphic to M_g and $A \simeq \oplus_{g \in G} M_g$; here δ_g is the characteristic function of the subset $\{g\}$.

• Let $g \in G$ such that δ_g is a primitive idempotent of A. Since A is Frobenius, $M_g \simeq A\delta_g$ has a unique simple submodule S and a unique maximal submodule N; M_g is the injective hull of S and the projective cover of M_g/N. See [CR, (9.9)].

• In all known cases, $\operatorname{gr} A \simeq \mathcal{B}(V)\#\Bbbk^G$, where V belongs to a concrete and short list. Hence, $\dim M_g = \dim \mathcal{B}(V)$ for any $g \in G$. More than this, in all known cases we dispose of the following information:

- ○ There exists a rack X and a 2-cocycle $q \in Z^2(X, \Bbbk^\times)$ such that $V \simeq (\Bbbk X, c^q)$ as braided vector spaces, see [AG] for details.
- ○ There exists an epimorphism of Hopf algebras $\phi : T(V)\#\Bbbk^G \to A$, see [AV, Subsection 2.5] for details. Note that $\phi(f \cdot x) = \operatorname{ad} f(\phi(x))$ for all $f \in \Bbbk^G$ and $x \in T(V)$.

- Let $\mathbb{X}$ be the set of words in X, identified with a basis of the tensor algebra $T(V)$. There exists $\mathbb{B} \subset \mathbb{X}$ such that the classes of the monomials in $\mathbb{B}$ form a basis of $\mathcal{B}(V)$. The corresponding classes in A multiplied with the elements $\delta_g \in \Bbbk^G$, $g \in G$, form a basis of A.
- If $x \in X$, then there exists $g_x \in G$ such that $\delta_h \cdot x = \delta_{h,g_x} x$ for all $h \in G$. We extend this to have $g_x \in G$ for any $x \in \mathbb{X}$.
- If $x \in X$, then $x^2 = 0$ in $\mathcal{B}(V)$ and there exists $f_x \in \Bbbk^G$ such that $x^2 = f_x$ in A.

Let $g \in G$. If $x \in \mathbb{B}$, then we denote by m_x the class of x in M_g. Hence $(m_x)_{x\in\mathbb{B}}$ is a basis of M_g. We may describe the action of A on this basis of M_g, at least when we know explicitly the relations of A and the monomials in $\mathbb{B}$. To start with, let $f \in \Bbbk^G$ and $x \in \mathbb{B}$. Then

$$\begin{aligned} f \cdot m_x = \overline{fx \otimes 1} = \overline{f_{(1)} \cdot x f_{(2)} \otimes 1} = \overline{f_{(1)} \cdot x \otimes f_{(2)} \cdot 1} \\ = f(g_x g)\, m_x. \end{aligned} \tag{1}$$

Let now $x = x_1 \dots x_t$ be a monomial in $\mathbb{B}$, with $x_1, \dots, x_t \in X$. Set $y = x_2 \dots x_t$; observe that y need not be in $\mathbb{B}$. Then

$$x_1 \cdot m_x = \overline{x_1^2 x_2 \dots x_t \otimes 1} = \overline{f_{x_1} y \otimes 1} = f_{x_1}(g_y g)\, \overline{y \otimes 1}. \tag{2}$$

Let now M be a finite dimensional A-module. It is convenient to consider the decomposition of M in isotypic components as $\Bbbk^G$-module: $M = \oplus_{g\in G} M[g]$, where $M[g] = \delta_g \cdot M$. Note that

$$x \cdot M[g] = M[g_x g] \qquad \text{for all } x \in \mathbb{B},\ g \in G. \tag{3}$$

For instance, (1) says that the isotypic components of the Verma module M_g are $M_g[h] = \langle m_x : x \in \mathbb{B},\ g_x g = h\rangle$.

2. Hopf algebras related to the class of transpositions in the symmetric group

2.1. Quadratic Nichols algebras.

Let $n \geq 3$; denote by $\mathcal{O}_2^n$ the conjugacy class of (12) in $\mathbb{S}_n$ and by $\operatorname{sgn} : C_{\mathbb{S}_n}(12) \to \Bbbk$ the restriction of the sign representation of $\mathbb{S}_n$ to the centralizer of (12). Let $V_n = M((12), \operatorname{sgn}) \in {}^{\Bbbk^{\mathbb{S}_n}}_{\Bbbk^{\mathbb{S}_n}}\mathcal{YD}$; V_n has a basis $(x_{(ij)})_{(ij)\in\mathcal{O}_2^n}$ such that the action $\cdot$ and the coaction δ are given by

$$\delta_h \cdot x_{(ij)} = \delta_{h,(ij)}\, x_{(ij)} \quad \forall h \in \mathbb{S}_n \quad \text{and} \quad \delta(x_{(ij)}) = \sum_{h\in\mathbb{S}_n} \operatorname{sgn}(h) \delta_h \otimes x_{h^{-1}(ij)h}.$$

Let $n = 3, 4, 5$. By [MS, G], we know that $\mathcal{B}(V_n)$ is quadratic and finite dimensional; actually, the ideal $\mathcal{J}_n$ of relations of $\mathcal{B}(V_n)$ is generated by

$$x_{(ij)}^2, \tag{4}$$

$$R_{(ij)(kl)} := x_{(ij)} x_{(kl)} + x_{(kl)} x_{(ij)}, \tag{5}$$

$$R_{(ij)(ik)} := x_{(ij)} x_{(ik)} + x_{(ik)} x_{(jk)} + x_{(jk)} x_{(ij)} \tag{6}$$

for $(ij), (kl), (ik) \in \mathcal{O}_2^n$ with $\#\{i, j, k, l\} = 4$.

For $n \geq 6$, we define the *quadratic Nichols algebra* $\mathcal{B}_n$ in the same way, that is as the quotient of the tensor algebra $T(V_n)$ by the ideal generated by the quadratic relations (4), (5) and (6) for $(ij), (kl), (ik) \in \mathcal{O}_2^n$ with $\#\{i, j, k, l\} = 4$. It is however open whether:

- $\mathcal{B}(V_n)$ is quadratic, i. e. isomorphic to $\mathcal{B}_n$;
- the dimension of $\mathcal{B}(V_n)$ is finite;
- the dimension of $\mathcal{B}_n$ is finite.

But we do know that the only possible finite dimensional Nichols algebras[2] over $\mathbb{S}_n$ are related to the orbit of transpositions and a pair of characters [AFGV, Th. 1.1]. Also, the Nichols algebras related to these two characters are twist-equivalent [Ve].

2.2. **The parameters.**

We consider the set of parameters

$$\mathfrak{A}_n := \Big\{ \mathbf{a} = (a_{(ij)})_{(ij)\in\mathcal{O}_2^n} \in \Bbbk^{\mathcal{O}_2^n} : \sum_{(ij)\in\mathcal{O}_2^n} a_{(ij)} = 0 \Big\}.$$

The group $\Gamma_n := \Bbbk^\times \times \operatorname{Aut}(\mathbb{S}_n)$ acts on $\mathfrak{A}_n$ by

$$(\mu, \theta) \triangleright \mathbf{a} = \mu(a_{\theta(ij)}), \qquad \mu \in \Bbbk^\times, \qquad \theta \in \operatorname{Aut}(\mathbb{S}_n), \qquad \mathbf{a} \in \mathfrak{A}_n. \tag{7}$$

Let $[\mathbf{a}] \in \Gamma_n \backslash \mathfrak{A}_n$ be the class of $\mathbf{a}$ under this action. Let $\triangleright$ denote also the conjugation action of $\mathbb{S}_n$ on itself, so that[3] $\mathbb{S}_n < \{e\} \times \operatorname{Aut}(\mathbb{S}_n) < \Gamma_n$. Let $\mathbb{S}_n^{\mathbf{a}} = \{g \in \mathbb{S}_n | g \triangleright \mathbf{a} = \mathbf{a}\}$ be the isotropy group of $\mathbf{a}$ under the action of $\mathbb{S}_n$.

We fix $\mathbf{a} \in \mathfrak{A}_n$ and introduce

$$f_{ij} = \sum_{g\in\mathbb{S}_n} (a_{(ij)} - a_{g^{-1}(ij)g})\delta_g \in \Bbbk^{\mathbb{S}_n}, \qquad (ij) \in \mathcal{O}_2^n. \tag{8}$$

Clearly,

$$f_{ij}(ts) = f_{ij}(s) \qquad \forall t \in C_{\mathbb{S}_n}(\mathrm{ij}), \quad s \in \mathbb{S}_n. \tag{9}$$

Definition 2. We say that g and $h \in \mathbb{S}_n$ are $\mathbf{a}$-*linked*, denoted $g \sim_{\mathbf{a}} h$, if either $g = h$ or else there exist $(i_m j_m), \dots, (i_1 j_1) \in \mathcal{O}_2^n$ such that

- $g = (i_m j_m) \cdots (i_1 j_1)h$,
- $f_{i_s j_s}((i_s j_s)(i_{s-1} j_{s-1}) \cdots (i_1 j_1)h) \neq 0$ for all $1 \leq s \leq m$.

In particular, $f_{i_1 j_1}(h) \neq 0$ by (9). We claim that $\sim_{\mathbf{a}}$ is an equivalence relation. For, if g and $h \in \mathbb{S}_n$ are $\mathbf{a}$-linked, then $h = (i_1 j_1) \cdots (i_m j_m)g$ and

[2]There is one exception when $n = 4$ that is finite dimensional and two exceptions when $n = 5$ and 6 that are not known.

[3]It is well-known that $\mathbb{S}_n$ identifies with the group of inner automorphisms and that this equals $\operatorname{Aut}\mathbb{S}_n$, except for $n = 6$.

$$f_{i_sj_s}((i_sj_s)(i_{s+1}j_{s+1})\cdots(i_mj_m)g) = f_{i_sj_s}((i_{s-1}j_{s-1})\cdots(i_1j_1)h)$$
$$\overset{(9)}{=} f_{i_sj_s}((i_sj_s)(i_{s-1}j_{s-1})\cdots(i_1j_1)h) \neq 0.$$

In the same way, we see that if $g \sim_{\mathbf{a}} h$ and also $h \sim_{\mathbf{a}} z$, then $g \sim_{\mathbf{a}} z$.

2.3. **A family of Hopf algebras.**

We fix $\mathbf{a} \in \mathfrak{A}_n$; recall the elements f_{ij} defined in (8). Let $\mathcal{I}_{\mathbf{a}}$ be the ideal of $T(V_n)\#\Bbbk^{\mathbb{S}_n}$ generated by (5), (6) and

$$x^2_{(ij)} - f_{ij}, \tag{10}$$

for all $(ij), (kl), (ik) \in \mathcal{O}^n_2$ such that $\#\{i, j, k, l\} = 4$. Then

$$\mathcal{A}_{[\mathbf{a}]} := T(V_n)\#\Bbbk^{\mathbb{S}_n}/\mathcal{I}_{\mathbf{a}}$$

is a Hopf algebra, see Remark 3. Also, if $\operatorname{gr}\mathcal{A}_{[\mathbf{a}]} \simeq \mathcal{B}(V_n)\#\Bbbk^{\mathbb{S}_n} \simeq \operatorname{gr}\mathcal{A}_{[\mathbf{b}]}$, then $\mathcal{A}_{[\mathbf{a}]} \simeq \mathcal{A}_{[\mathbf{b}]}$ if and only if $[\mathbf{a}] = [\mathbf{b}]$, what justifies the notation. If $n = 3$, then $\operatorname{gr}\mathcal{A}_{[\mathbf{a}]} \simeq \mathcal{B}(V_3)\#\Bbbk^{\mathbb{S}_3}$ and $\dim \mathcal{A}_{[\mathbf{a}]} = 72$ [AV]; for $n = 4, 5$ the dimension is finite but we do not know if it is the "right" one; for $n \geq 6$, the dimension is unknown to be finite.

Remark 3. A straightforward computation shows that

$$\Delta(x^2_{(ij)}) = x^2_{(ij)}\otimes 1 + \sum_{h\in\mathbb{S}_n} \delta_h\otimes x^2_{h^{-1}(ij)h} \quad \text{and}$$
$$\Delta(f_{ij}) = f_{ij}\otimes 1 + \sum_{h\in\mathbb{S}_n} \delta_h\otimes f_{h^{-1}(i)h^{-1}(j)}.$$

Then $J = \langle x^2_{(ij)} - f_{ij} : (ij) \in \mathcal{O}^n_2\rangle$ is a coideal. Since $f_{ij}(e) = 0$, we have that $J \subset \ker\epsilon$ and $\mathcal{S}(J) \subseteq \Bbbk^{\mathbb{S}_n}J$. Thus $\mathcal{I}_{\mathbf{a}} = (J)$ is a Hopf ideal and $\mathcal{A}_{[\mathbf{a}]}$ is a Hopf algebra quotient of $T(V_n)\#\Bbbk^{\mathbb{S}_n}$. We shall say that $\Bbbk^{\mathbb{S}_n}$ *is a subalgebra of* $\mathcal{A}_{[\mathbf{a}]}$ to express that the restriction of the projection $T(V_n)\#\Bbbk^{\mathbb{S}_n} \twoheadrightarrow \mathcal{A}_{[\mathbf{a}]}$ to $\Bbbk^{\mathbb{S}_n}$ is injective.

Let us collect a few general facts on the representation theory of $\mathcal{A}_{[\mathbf{a}]}$.

Remark 4. Assume that $\Bbbk^{\mathbb{S}_n}$ is a subalgebra of $\mathcal{A}_{[\mathbf{a}]}$ and let M be an $\mathcal{A}_{[\mathbf{a}]}$-module. Hence

(a) If $(ij) \in \mathcal{O}^n_2$ satisfies $f_{ij}(h) \neq 0$, then $\rho(x_{(ij)}) : M[h] \to M[\text{(ij)}h]$ is an isomorphism.

(b) Let $g \sim_{\mathbf{a}} h \in \mathbb{S}_n$. Then $\rho(x_{(i_mj_m)}) \circ \cdots \circ \rho(x_{(i_1j_1)}) : M[h] \to M[g]$ is an isomorphism.

Proof. $\rho(x_{(ij)}) : M[h] \to M[\text{(ij)}h]$ is injective and $\rho(x_{(ij)}) : M[\text{(ij)}h] \to M[h]$ is surjective, by (10). Interchanging the roles of h and $\text{(ij)}h$, we get (a). Now (b) follows from (a). $\square$

This Remark is particularly useful to compare Verma modules.

Proposition 5. *Assume that* $\dim \mathcal{A}_{[\mathbf{a}]} < \infty$ *and* $\Bbbk^{\mathbb{S}_n}$ *is a subalgebra of* $\mathcal{A}_{[\mathbf{a}]}$. *If* g *and* h *are* $\mathbf{a}$*-linked, then the Verma modules* M_g *and* M_h *are isomorphic.*

Proof. The Verma module M_h is generated by $m_1 = 1 \otimes_{\Bbbk^{\mathbb{S}_n}} 1 \in M_h[h]$. By Remark 4 (b), there exists $m \in M_h[g]$ such that $M_h = \mathcal{A}_{[\mathbf{a}]} \cdot m$. Therefore, there is an epimorphism $M_g \twoheadrightarrow M_h$. Since $\mathcal{A}_{[\mathbf{a}]}$ is finite dimensional, all the Verma modules have the same dimension; hence $M_g \simeq M_h$. □

Definition 6. We say that the parameter $\mathbf{a}$ is *generic* when any of the following equivalent conditions holds.

(a) $a_{(ij)} \neq a_{(kl)}$ for all $(ij) \neq (kl) \in \mathcal{O}_2^n$.
(b) $a_{(ij)} \neq a_{h \triangleright (ij)}$ for all $(ij) \in \mathcal{O}_2^n$ and all $h \in \mathbb{S}_n - C_{\mathbb{S}_n}(ij)$.
(c) $f_{ij}(h) \neq 0$ for all $(ij) \in \mathcal{O}_2^n$ and all $h \in \mathbb{S}_n - C_{\mathbb{S}_n}(ij)$.

Proof. (a) $\implies$ (b) is clear, since $(ij) \neq h \triangleright (ij)$ by the assumption on h. (b) $\implies$ (a) follows since any $(kl) \neq (ij)$ is of the form $(kl) = h \triangleright (ij)$, for some $h \notin \mathbb{S}_n^{(ij)}$. (b) $\iff$ (c): given (ij), we have

$$\{h \in \mathbb{S}_n : a_{(ij)} = a_{h \triangleright (ij)}\} = \{h \in \mathbb{S}_n : f_{ij}(h) = 0\};$$

hence, one of these sets equals $C_{\mathbb{S}_n}(ij)$ iff the other does. □

Lemma 7. *Assume that* $\mathbf{a}$ *is generic, so that* $g \sim_{\mathbf{a}} h$ *for all* $g, h \in \mathbb{S}_n - \{e\}$. *If* $\Bbbk^{\mathbb{S}_n}$ *is a subalgebra of* $\mathcal{A}_{[\mathbf{a}]}$, *then*

(a) *If* $\mathcal{A}_{[\mathbf{a}]}$ *is finite dimensional, then the Verma modules* M_g *and* M_h *are isomorphic, for all* $g, h \in \mathbb{S}_n - \{e\}$.
(b) *If* M *is an* $\mathcal{A}_{[\mathbf{a}]}$*-module, then* $\dim M[h] = \dim M[g]$ *for all* $g, h \in \mathbb{S}_n - \{e\}$. *Thus* $\dim M = (n! - 1) \dim M[(ij)] + \dim M[e]$.
(c) *If* M *is simple and* $n = 3$, *then* $\dim M[h] \leq 1$ *for all* $h \in \mathbb{S}_3 - \{e\}$.

Proof. Let $(ij) \in \mathbb{S}_n$ and $g \in \mathbb{S}_n - \{e\}$.

- If $g = (ik)$, then $g \sim_{\mathbf{a}} (ij)$, as $(ik) = (jk)(ij)(jk)$ and $\mathbf{a}$ is generic.
- If $g = (kl)$ with $\#\{i, j, l, k\} = 4$, then $(ij) \sim_{\mathbf{a}} (ik)$ and $(ik) \sim_{\mathbf{a}} (kl)$, hence $(ij) \sim_{\mathbf{a}} (kl)$.
- If $g = (i_1 i_2 \cdots i_r)$ is an r-cycle, then $g = (i_1 i_r)(i_1 i_2 \cdots i_{r-1})$. Hence $g \sim_{\mathbf{a}} (ij)$ by induction on r.
- Let $g = g_1 \cdots g_m$ be the product of the disjoint cycles $g_1, \dots, g_m$, with $m \geq 2$; say $g_1 = (i_1 \cdots i_r)$, $g_2 = (i_{r+1} \cdots i_{r+s})$ and denote $y = g_3 \cdots g_m$. Then $g = (i_1 i_{r+1})(i_1 \cdots i_{r+s})y$ and $y \in C_{\mathbb{S}_n}(i_1 i_{r+1})$. Hence g and (ij) are linked by induction on m.

Now (a) follows from Proposition 5 and (b) from Remark 4. If $n = 3$ and M is simple, then $\dim \mathcal{A}_{[\mathbf{a}]} = 72 > (\dim M)^2 \geq 25(\dim M[(12)])^2$ and the last assertion of the lemma follows. □

The characterization of all one dimensional $\mathcal{A}_{[\mathbf{a}]}$-modules is not difficult. Let $\approx$ be the equivalence relation in $\mathcal{O}_2^n$ given by $(ij) \approx (kl)$ iff $a_{(ij)} = a_{(kl)}$.

Let $\mathcal{O}_2^n = \coprod_{s\in\Upsilon} \mathcal{C}_s$ be the associated partition. If $h \in \mathbb{S}_n$, then

$$f_{ij}(h) = 0\,\forall (ij) \in \mathcal{O}_2^n \iff h^{-1}\mathcal{C}_s h = \mathcal{C}_s\,\forall s \in \Upsilon \iff h \in \mathbb{S}_n^{\mathbf{a}}. \tag{11}$$

Lemma 8. *Assume that $\Bbbk^{\mathbb{S}_n}$ is a subalgebra of $\mathcal{A}_{[\mathbf{a}]}$ and let $h \in \mathbb{S}_n^{\mathbf{a}}$. Then $\Bbbk_h$ is a $\mathcal{A}_{[\mathbf{a}]}$-module with the action given by the algebra map $\zeta_h : \mathcal{A}_{[\mathbf{a}]} \to \Bbbk$,*

$$\zeta_h(x_{(ij)}) = 0, \quad (ij) \in \mathcal{O}_2^n \quad \text{and} \quad \zeta_h(f) = f(h), \quad f \in \Bbbk^{\mathbb{S}_n}. \tag{12}$$

The one-dimensional representations of $\mathcal{A}_{[\mathbf{a}]}$ are all of this form.

Proof. Clearly, ζ_h satisfies the relations of $T(V_n)\#\Bbbk^{\mathbb{S}_n}$, (5) and (6); (10) holds because h fulfills (11). Now, let M be a module of dimension 1. Then $M = M[h]$ for some h; thus $f_{ij}(h) = 0$ for all $(ij) \in \mathcal{O}_2^n$ by Remark 4. □

3. Simple and Verma modules over Hopf algebras with coradical $\Bbbk^{\mathbb{S}_3}$

3.1. Verma modules.

In this Section, we focus on the case $n = 3$. Let $\mathbf{a} \in \mathfrak{A}_3$. Explicitly, $\mathcal{A}_{[\mathbf{a}]}$ is the algebra $(T(V_3)\#\Bbbk^{\mathbb{S}_3})/\mathcal{I}_{\mathbf{a}}$ where $\mathcal{I}_{\mathbf{a}}$ is the ideal generated by

$$R_{(13)(23)}, \qquad R_{(23)(13)}, \qquad x_{(ij)}^2 - f_{ij}, \qquad (ij) \in \mathcal{O}_2^3, \tag{13}$$

where

$$\begin{aligned} f_{13} &= (a_{(13)} - a_{(23)})(\delta_{(12)} + \delta_{(123)}) + (a_{(13)} - a_{(12)})(\delta_{(23)} + \delta_{(132)}),\\ f_{23} &= (a_{(23)} - a_{(12)})(\delta_{(13)} + \delta_{(123)}) + (a_{(23)} - a_{(13)})(\delta_{(12)} + \delta_{(132)}),\\ f_{12} &= (a_{(12)} - a_{(13)})(\delta_{(23)} + \delta_{(123)}) + (a_{(12)} - a_{(23)})(\delta_{(13)} + \delta_{(132)}). \end{aligned} \tag{14}$$

We know from [AV] that $\mathcal{A}_{[\mathbf{a}]}$ is a Hopf algebra of dimension 72 and coradical isomorphic to $\Bbbk^{\mathbb{S}_3}$, for any $\mathbf{a} \in \mathfrak{A}_3$. Furthermore, any finite dimensional non-semisimple Hopf algebra with coradical $\Bbbk^{\mathbb{S}_3}$ is isomorphic to $\mathcal{A}_{[\mathbf{a}]}$ for some $\mathbf{a} \in \mathfrak{A}_3$; $\mathcal{A}_{[\mathbf{b}]} \simeq \mathcal{A}_{[\mathbf{a}]}$ iff $[\mathbf{a}] = [\mathbf{b}]$. Let $\Omega = f_{13}((12)\underline{\quad}) - f_{13}$, that is

$$\begin{aligned} \Omega =&(a_{(23)} - a_{(13)})(\delta_{(12)} - \delta_e)\\ &+ (a_{(13)} - a_{(12)})(\delta_{(13)} - \delta_{(132)}) + (a_{(12)} - a_{(23)})(\delta_{(23)} - \delta_{(123)}). \end{aligned} \tag{15}$$

The following formulae follow from the defining relations:

$$x_{(12)}x_{(13)}x_{(12)} = x_{(13)}x_{(12)}x_{(13)} + x_{(23)}(a_{(13)} - a_{(12)}), \tag{16}$$

$$x_{(23)}x_{(12)}x_{(23)} = x_{(12)}x_{(23)}x_{(12)} - x_{(13)}(a_{(23)} - a_{(12)}) \text{ and} \tag{17}$$

$$x_{(23)}x_{(12)}x_{(13)} = x_{(13)}x_{(12)}x_{(23)} + x_{(12)}\Omega. \tag{18}$$

Let

$$\mathbb{B} = \left\{ \begin{matrix} 1, & x_{(13)}, & x_{(13)}x_{(12)}, & x_{(13)}x_{(12)}x_{(13)}, & x_{(13)}x_{(12)}x_{(23)}x_{(12)},\\ & x_{(23)}, & x_{(12)}x_{(13)}, & x_{(12)}x_{(23)}x_{(12)}, & \\ & x_{(12)}, & x_{(23)}x_{(12)}, & x_{(13)}x_{(12)}x_{(23)}, & \\ & & x_{(12)}x_{(23)} & & \end{matrix} \right\}.$$

Then $\{x\delta_g | x \in \mathbb{B},\, g \in \mathbb{S}_3\}$ is a basis of $\mathcal{A}_{[\mathbf{a}]}$ [AV]. Fix $g \in G$. The classes of the monomials in $\mathbb{B}$ form a basis of the Verma module M_g. Denote by

$m_{(ij)\dots(rs)}$ the class of $x_{(ij)}\dots x_{(rs)}$; we simply set $m_{\mathsf{top}} = m_{(13)(12)(23)(12)}$. The action of $\mathcal{A}_{[\mathbf{a}]}$ on M_g is described in this basis by the following formulae:

$$f \cdot m_1 = f(g)m_1, \qquad f \in \Bbbk^{\mathbb{S}_3}; \tag{19}$$

$$f \cdot m_{(ij)\dots(rs)} = f\big(\mathrm{(ij)}\dots\mathrm{(rs)}g\big)\, m_{(ij)\dots(rs)}, \qquad f \in \Bbbk^{\mathbb{S}_3}; \tag{20}$$

$$x_{(ij)} \cdot m_1 = m_{(ij)}, \qquad (ij) \in \mathcal{O}_2^3; \tag{21}$$

$$x_{(ij)} \cdot m_{(ij)} = f_{ij}(g)m_1, \qquad (ij) \in \mathcal{O}_2^3; \tag{22}$$

$$x_{(13)} \cdot m_{(23)} = -m_{(23)(12)} - m_{(12)(13)}, \tag{23}$$

$$x_{(13)} \cdot m_{(12)} = m_{(13)(12)}, \tag{24}$$

$$x_{(23)} \cdot m_{(13)} = -m_{(12)(23)} - m_{(13)(12)}, \tag{25}$$

$$x_{(23)} \cdot m_{(12)} = m_{(23)(12)}, \tag{26}$$

$$x_{(12)} \cdot m_{(13)} = m_{(12)(13)}, \tag{27}$$

$$x_{(12)} \cdot m_{(23)} = m_{(12)(23)}; \tag{28}$$

$$x_{(13)} \cdot m_{(13)(12)} = f_{13}\big((12)g\big)\, m_{(12)}, \tag{29}$$

$$x_{(13)} \cdot m_{(12)(13)} = m_{(13)(12)(13)}, \tag{30}$$

$$x_{(13)} \cdot m_{(23)(12)} = -m_{(13)(12)(13)} - f_{13}\big((23)g\big)\, m_{(23)} \tag{31}$$

$$x_{(13)} \cdot m_{(12)(23)} = m_{(13)(12)(23)}; \tag{32}$$

$$x_{(23)} \cdot m_{(13)(12)} = -m_{(12)(23)(12)} - f_{12}(g)m_{(13)}, \tag{33}$$

$$x_{(23)} \cdot m_{(12)(13)} = m_{(13)(12)(23)} + \Omega(g)m_{(12)}, \tag{34}$$

$$x_{(23)} \cdot m_{(23)(12)} = f_{23}\big((12)g\big)m_{(12)}, \tag{35}$$

$$x_{(23)} \cdot m_{(12)(23)} = m_{(12)(23)(12)} - m_{(13)}f_{23}\big((13)\big), \tag{36}$$

$$x_{(12)} \cdot m_{(13)(12)} = m_{(13)(12)(13)} + m_{(23)}f_{13}\big((23)\big), \tag{37}$$

$$x_{(12)} \cdot m_{(12)(13)} = f_{12}\big((13)g\big)m_{(13)}, \tag{38}$$

$$x_{(12)} \cdot m_{(23)(12)} = m_{(12)(23)(12)}, \tag{39}$$

$$x_{(12)} \cdot m_{(12)(23)} = f_{12}\big((23)g\big)m_{(23)}; \tag{40}$$

$$x_{(13)} \cdot m_{(13)(12)(13)} = f_{13}\big((12)(13)g\big)\, m_{(12)(13)}, \tag{41}$$

$$x_{(13)} \cdot m_{(12)(23)(12)} = m_{\mathsf{top}}, \tag{42}$$

$$x_{(13)} \cdot m_{(13)(12)(23)} = f_{13}\big((12)(23)g\big)\, m_{(12)(23)}, \tag{43}$$

$$x_{(23)} \cdot m_{(13)(12)(13)} = m_{\mathsf{top}} - \big(f_{12}\Omega + (a_{(13)} - a_{(12)})f_{23}\big)(g)m_1, \tag{44}$$

$$x_{(23)} \cdot m_{(12)(23)(12)} = f_{12}(g)m_{(12)(23)} + (a_{(12)} - a_{(23)})m_{(13)(12)}, \tag{45}$$

$$x_{(23)} \cdot m_{(13)(12)(23)} = f_{23}\big((23)(12)g\big)m_{(12)(13)} - \Omega(g)m_{(23)(12)}, \tag{46}$$

$$x_{(12)} \cdot m_{(13)(12)(13)} = \big(f_{13}(g) + f_{12}((23))\big)m_{(13)(12)} + f_{12}\big((23)\big)m_{(12)(23)}, \tag{47}$$

$$x_{(12)} \cdot m_{(12)(23)(12)} = f_{12}\big((23)(12)g\big)\, m_{(23)(12)}, \tag{48}$$

$$x_{(12)} \cdot m_{(13)(12)(23)} = -m_{\mathsf{top}} + \big(f_{13}((23))f_{23} - f_{12}((13)\underline{\quad})f_{13}\big)(g)m_1; \tag{49}$$

$$x_{(13)} \cdot m_{\mathsf{top}} = f_{13}(g)\, m_{(12)(23)(12)}, \tag{50}$$

$$x_{(23)} \cdot m_{\mathsf{top}} = f_{23}(g) m_{(13)(12)(13)} + (f_{13}((23)) f_{23} + \Omega f_{12})(g) m_{(23)}, \tag{51}$$

$$\begin{aligned} x_{(12)} \cdot m_{\mathsf{top}} = &-f_{12}(g) m_{(13)(12)(23)} \\ &+ (f_{13}((23)) f_{23}((12)\_\_) - f_{12}((23)\_\_) f_{13}((12)\_\_))(g) m_{(12)}; \end{aligned} \tag{52}$$

To proceed with the description of the simple modules, we split the consideration of the algebras $\mathcal{A}_{[\mathbf{a}]}$ into several cases.

- $a_{(13)} = a_{(12)} = a_{(23)}$. In this case, there is a projection $\mathcal{A}_{[\mathbf{a}]} \to \Bbbk^{\mathbb{S}_3}$. It is easy to see that any simple $\mathcal{A}_{[\mathbf{a}]}$-module is obtained from a simple $\Bbbk^{\mathbb{S}_3}$-module composing with this projection; thus, $\widehat{\mathcal{A}_{[\mathbf{a}]}} \simeq \mathbb{S}_3$.
- $a_{(13)} = a_{(12)}$ or $a_{(23)} = a_{(12)}$ or $a_{(13)} = a_{(23)}$, but not in the previous case. Up to isomorphism, cf. (7), we may assume $a_{(12)} \neq a_{(13)} = a_{(23)}$. For shortness, we shall say that $\mathbf{a}$ is *sub-generic*.
- $\mathbf{a}$ is generic.

In the next subsections, we investigate these two different cases. Let us consider the decomposition of the Verma module M_g in isotypic components as $\Bbbk^{\mathbb{S}_3}$-modules. The isotypic components of the Verma module M_e are

$$\begin{aligned} M_e[e] &= \langle m_1, m_{\mathsf{top}} \rangle, & M_e[(12)] &= \langle m_{(12)}, m_{(13)(12)(23)} \rangle, \\ M_e[(13)] &= \langle m_{(13)}, m_{(12)(23)(12)} \rangle, & M_e[(23)] &= \langle m_{(23)}, m_{(13)(12)(13)} \rangle, \\ M_e[(123)] &= \langle m_{(13)(12)}, m_{(12)(23)} \rangle, & M_e[(132)] &= \langle m_{(12)(13)}, m_{(23)(12)} \rangle. \end{aligned} \tag{53}$$

Let $g, h \in \mathbb{S}_3$, $(ij) \in \mathcal{O}_2^3$. By (20) and (3), we have

$$M_g[h] = M_e[hg^{-1}], \tag{54}$$

$$x_{(ij)} \cdot M_g[h] \subseteq M_g[(\mathrm{ij})h]. \tag{55}$$

It is convenient to introduce the following elements:

$$m_{\mathsf{soc}} = f_{13}((23)) f_{23}((13)) m_1 - m_{\mathsf{top}}, \tag{56}$$

$$m_{\mathsf{o}} = m_{(13)(12)(13)} + f_{13}((23)) m_{(23)}. \tag{57}$$

3.2. **Case $\mathbf{a} \in \mathfrak{A}_3$ generic.**

To determine the simple $\mathcal{A}_{[\mathbf{a}]}$-modules, we just need to determine the maximal submodules of the various Verma modules. By Lemma 7 (a), we are reduced to consider the Verma modules M_e and M_g for some fixed $g \neq e$. We choose $g = (13)(23)$; for the sake of an easy exposition, we write the elements of $\mathbb{S}_3$ as products of transpositions.

We start with the following observation. Let M be a cyclic $\mathcal{A}_{[\mathbf{a}]}$-module, generated by $v \in M[(13)(23)]$. By (55) and acting by the monomials in our basis of $\mathcal{A}_{[\mathbf{a}]}$, we see that $M[(23)(13)] = \langle x_{(13)} x_{(23)} \cdot v, x_{(23)} x_{(12)} \cdot v, x_{(12)} x_{(13)} \cdot v \rangle$. This weight space is $\neq 0$ by Lemma 7 (b), and a further application of this Lemma gives the following result.

Remark 9. Let M be a cyclic $\mathcal{A}_{[\mathbf{a}]}$-module, generated by $v \in M[(13)(23)]$. If $\dim M[(23)(13)] = 1$, then

$$\begin{aligned} M[(23)] &= \langle x_{(13)} \cdot v\rangle, & M[e] &= \langle x_{(12)}x_{(23)} \cdot v, x_{(13)}x_{(12)} \cdot v\rangle, \\ M[(12)] &= \langle x_{(23)} \cdot v\rangle, & M[(13)] &= \langle x_{(12)} \cdot v\rangle, \\ M[(13)(23)] &= \langle v\rangle, & M[(23)(13)] &= \langle x_{(13)}x_{(23)} \cdot v\rangle. \end{aligned} \tag{58}$$

Thus, any cyclic module as in the Remark has either dimension 5, 6 or 7. Moreover, there is a simple module L like this; L has a basis $\{v_g | e \neq g \in \mathbb{S}_3\}$ and the action is given by

$$v_g \in L[g], \qquad x_{(ij)} \cdot v_g = \begin{cases} v_{(ij)g} & \text{if } \operatorname{sgn} g = 1, \\ f_{ij}(g)v_{(ij)g} & \text{if } \operatorname{sgn} g = -1. \end{cases} \tag{59}$$

Let $\mathbb{k}_e$ be as in Lemma 8. We shall see that L and $\mathbb{k}_e$ are the only simple modules of $\mathcal{A}_{[\mathbf{a}]}$.

The Verma module M_e projects onto the simple submodule $\mathbb{k}_e$, hence the kernel of this projection is a maximal submodule; explicitly this is

$$N_e = \mathcal{A}_{[\mathbf{a}]} \cdot M_e[(13)(23)] = \oplus_{g\sim_{\mathbf{a}}(13)(23)} M_e[g] \oplus \langle m_{\mathsf{top}}\rangle.$$

We see that this is the unique maximal submodule, as consequence of the following description of all submodules of M_e.

Lemma 10. *The submodules of M_e are*

$$\langle m_{top}\rangle \subsetneq \mathcal{A}_{[\mathbf{a}]} \cdot v \subsetneq N_e \subsetneq M_e$$

for any $v \in M_e[(13)(23)] - 0$. The submodules $\mathcal{A}_{[\mathbf{a}]} \cdot v$ and $\mathcal{A}_{[\mathbf{a}]} \cdot u$ coincide iff $v \in \langle u\rangle$. The quotients $\mathcal{A}_{[\mathbf{a}]} \cdot v/\langle m_{top}\rangle$ and $N_e/\mathcal{A}_{[\mathbf{a}]} \cdot v$ are isomorphic to L; and M_e/N_e and $\langle m_{top}\rangle$ are isomorphic to $\mathbb{k}_e$.

Proof. By (51), (50) and (52), we have $x_{(ij)} \cdot m_{\mathsf{top}} = 0$ for all $(ij) \in \mathcal{O}_2^3$. Let

$$\begin{aligned} v &= \lambda m_{(23)(12)} + \mu m_{(12)(13)} & &\in M_e[(13)(23)] - 0, \\ w &= \mu m_{(12)(23)} + (\mu - \lambda) m_{(13)(12)} & &\in M_e[(23)(13)]. \end{aligned}$$

Using the formulae (23) to (49), we see that $x_{(13)}x_{(23)} \cdot v$, $x_{(23)}x_{(12)} \cdot v$ and $x_{(12)}x_{(13)} \cdot v$ are non-zero multiples of w. That is, $\dim(\mathcal{A}_{[\mathbf{a}]} \cdot v)[(23)(13)] = 1$. Also, $x_{(12)}x_{(23)} \cdot v = -\mu m_{\mathsf{top}}$ and $x_{(13)}x_{(12)} \cdot v = \lambda m_{\mathsf{top}}$. Hence

$$\Big\{v,\, x_{(23)} \cdot v,\, x_{(12)} \cdot v,\, x_{(13)} \cdot v,\, w,\, m_{\mathsf{top}}\Big\}$$

is a basis of $\mathcal{A}_{[\mathbf{a}]} \cdot v$ by Remark 9.

Let now N be a (proper, non-trivial) submodule of M_e. If $N \neq \langle m_{\mathsf{top}}\rangle$, then there exists $v \in N[(13)(23)] - 0$. Hence $\mathcal{A}_{[\mathbf{a}]} \cdot v$ is a submodule of N and $N[e] = \langle m_{\mathsf{top}}\rangle$ because $m_1 \in M_e[e]$ and $\dim M_e[e] = 2$. Therefore $N = \mathcal{A}_{[\mathbf{a}]} \cdot N[(13)(23)]$. □

It is convenient to introduce the following $\mathcal{A}_{[\mathbf{a}]}$-modules which we will use in the Section 4.

Definition 11. Let $\mathbf{t} \in \mathfrak{A}_3$. We denote by $W_{\mathbf{t}}(L, \Bbbk_e)$ the $\mathcal{A}_{[\mathbf{a}]}$-module with basis $\{w_g : g \in \mathbb{S}_3\}$ and action given by

$$w_g \in W_{\mathbf{t}}(L, \Bbbk_e)[g], \quad x_{(ij)} \cdot w_g = \begin{cases} 0 & \text{if } g = e, \\ w_{(ij)g} & \text{if } g \neq e \text{ and } \operatorname{sgn} g = 1, \\ f_{ij}(g) w_{(ij)g} & \text{if } g \neq (ij) \text{ and } \operatorname{sgn} g = -1, \\ t_{(ij)} w_e & \text{if } g = (ij). \end{cases}$$

The well-definition of $W_{\mathbf{t}}$ follows from the next lemma.

Lemma 12. *Let* $\mathbf{t}, \tilde{\mathbf{t}} \in \mathfrak{A}_3$.

(a) *If* $\mathbf{t} = (0,0,0)$, *then* $W_{\mathbf{t}}(L, \Bbbk_e) \simeq \Bbbk_e \oplus L$.

(b) *If* $\mathbf{t} \neq (0,0,0)$, *then there exists* $v \in M_e[(13)(23)] - 0$ *such that* $W_{\mathbf{t}}(L, \Bbbk_e) \simeq \mathcal{A}_{[\mathbf{a}]} \cdot v$.

(c) *If* $v \in M_e[(13)(23)] - 0$, *then there exists* $\mathbf{t} \neq (0,0,0)$ *such that* $W_{\mathbf{t}}(L, \Bbbk_e) \simeq \mathcal{A}_{[\mathbf{a}]} \cdot v$.

(d) $W_{\mathbf{t}}(L, \Bbbk_e)$ *is an extension of* L *by* $\Bbbk_e$.

(e) $W_{\mathbf{t}}(L, \Bbbk_e) \simeq W_{\tilde{\mathbf{t}}}(L, \Bbbk_e)$ *if and only if* $\mathbf{t} = \mu \tilde{\mathbf{t}}$ *with* $\mu \in \Bbbk^{\times}$.

Proof. (a) is immediate. If we prove (b), then (d) follows from Lemma 10.

(b) We set $w_{(13)(23)} = t_{(13)} m_{(23)(12)} - t_{(12)} m_{(12)(13)} \in M_e[(13)(23)] - 0$,

$$w_{(23)} = x_{(13)} \cdot w_{(13)(23)}, \quad w_{(13)} = x_{(12)} \cdot w_{(13)(23)}, \quad w_{(12)} = x_{(23)} \cdot w_{(13)(23)},$$

$$w_{(23)(13)} = \frac{1}{f_{23}((13))} x_{(23)} x_{(12)} \cdot w_{(13)(23)} \quad \text{and} \quad w_e = m_{\mathsf{top}}.$$

By the proof of Lemma 10 and (17), we see that $W_{\mathbf{t}}(L, \Bbbk_e) \simeq \mathcal{A}_{[\mathbf{a}]} \cdot w_{(13)(23)}$. (c) follows from the proof of Lemma 10. (e) Let $\{\tilde{w}_g : g \in \mathbb{S}_3\}$ be the basis of $W_{\tilde{\mathbf{t}}}(L, \Bbbk_e)$ according to Definition 11. Let $F : W_{\mathbf{t}}(L, \Bbbk_e) \to W_{\tilde{\mathbf{t}}}(L, \Bbbk_e)$ be an isomorphism of $\mathcal{A}_{[\mathbf{a}]}$-module. Since F is an isomorphism of $\Bbbk^{\mathbb{S}_3}$-modules, there exists $\mu_g \in \Bbbk^{\times}$ for all $g \in \mathbb{S}_3$ such that $F(w_g) = \mu_g \tilde{w}_g$. In particular, F induces an automorphism of L. Since L is simple (cf. Theorem 1), $\mu_g = \mu_L$ for all $g \neq e$. Since $F(x_{(ij)} \cdot w_{(ij)}) = x_{(ij)} \cdot F(w_{(ij)})$, we see that $\mathbf{t} = \frac{\mu_L}{\mu_e} \tilde{\mathbf{t}}$. Conversely, F is well defined for all μ_e and μ_L such that $\mu = \frac{\mu_L}{\mu_e}$. $\square$

The Verma module $M_{(13)(23)}$ projects onto the simple module L, hence the kernel of this projection is a maximal submodule; explicitly this is

$$N_{(13)(23)} = \mathcal{A}_{[\mathbf{a}]} \cdot M_{(13)(23)}[e] = M_{(13)(23)}[e] \oplus \mathcal{A}_{[\mathbf{a}]} \cdot m_{\mathsf{soc}}.$$

We see that this is the unique maximal submodule, as consequence of the following description of all submodules of $M_{(13)(23)}$. Recall m_{soc} from (56).

Lemma 13. *The submodules of* $M_{(13)(23)}$ *are*

$$\mathcal{A}_{[\mathbf{a}]} \cdot m_{\mathsf{soc}} \subsetneq \mathcal{A}_{[\mathbf{a}]} \cdot v \subsetneq N_{(13)(23)} \subsetneq M_{(13)(23)}$$

for all $v \in M_{(13)(23)}[e] - 0$. *The submodules* $\mathcal{A}_{[\mathbf{a}]} \cdot v$ *and* $\mathcal{A}_{[\mathbf{a}]} \cdot u$ *coincide iff* $v \in \langle u \rangle$. *The quotients* $\mathcal{A}_{[\mathbf{a}]} \cdot v / \mathcal{A}_{[\mathbf{a}]} \cdot m_{\mathsf{soc}}$ *and* $N_{(13)(23)} / \mathcal{A}_{[\mathbf{a}]} \cdot v$ *are isomorphic to* $\Bbbk_e$; *and* $M_{(13)(23)} / N_{(13)(23)}$ *and* $\mathcal{A}_{[\mathbf{a}]} \cdot m_{\mathsf{soc}}$ *are isomorphic to* L.

Proof. Let $v = \lambda m_1 + \mu m_{\mathsf{top}} \in M_{(13)(23)}[(13)(23)] - 0$ and $N = \mathcal{A}_{[\mathbf{a}]} \cdot v$. Using the formulae (23) to (49), we see that

$$x_{(12)}x_{(13)} \cdot v = \lambda m_{(12)(13)} - \mu f_{13}((23))^2 m_{(23)(12)} \quad \text{and}$$
$$x_{(23)}x_{(12)} \cdot v = \mu f_{23}((13))^2 m_{(12)(13)} + \big(\lambda + 2\mu f_{13}((23)) f_{23}((13))\big) m_{(23)(12)}.$$

Thus, $\dim N[(23)(13)] = 1$ iff $\lambda + \mu f_{13}((23)) f_{23}((13)) = 0$, that is iff $v \in \langle m_{\mathsf{soc}} \rangle - 0$. In this case,

$$\Big\{ v,\, x_{(23)} \cdot v,\, x_{(12)} \cdot v,\, x_{(13)} \cdot v,\, x_{(12)}x_{(13)} \cdot v \Big\}$$

is a basis of $\mathcal{A}_{[\mathbf{a}]} \cdot m_{\mathsf{soc}}$ by Remark 9.

Let now N be an arbitrary submodule of $M_{(13)(23)}$. If $\dim N[(13)(23)] = 2$, then $N = M_{(13)(23)}$. If $\dim N[(13)(23)] = 0$, then $N \subset M_{(13)(23)}[e]$ by Lemma 7. But this is not possible since $\ker x_{(13)} \cap \ker x_{(23)} \cap \ker x_{(12)} = 0$, what is checked using the formulae (23) to (52). It remains the case $\dim N[(13)(23)] = 1$. By the argument at the beginning of the proof, the lemma follows. $\square$

It is convenient to introduce the following $\mathcal{A}_{[\mathbf{a}]}$-modules which we will use in the Section 4.

Definition 14. Let $\mathbf{t} \in \mathfrak{A}_3$. We denote by $W_{\mathbf{t}}(\Bbbk_e, L)$ the $\mathcal{A}_{[\mathbf{a}]}$-module with basis $\{w_g : g \in \mathbb{S}_3\}$ and action given by

$$w_g \in W_{\mathbf{t}}(\Bbbk_e, L)[g], \quad x_{(ij)} \cdot w_g = \begin{cases} t_{(ij)} w_{(ij)} & \text{if } g = e, \\ f_{ij}(g) w_{(ij)g} & \text{if } g \neq e \text{ and } \operatorname{sgn} g = 1, \\ w_{(ij)g} & \text{if } \operatorname{sgn} g = -1. \end{cases}$$

The well-definition of $W_{\mathbf{t}}(\Bbbk_e, L)$ follows from the next lemma.

Lemma 15. *Let* $\mathbf{t}, \tilde{\mathbf{t}} \in \mathfrak{A}_3$.

(a) *If* $\mathbf{t} = (0,0,0)$, *then* $W_{\mathbf{t}}(\Bbbk_e, L) \simeq L \oplus \Bbbk_e$.

(b) *If* $\mathbf{t} \neq (0,0,0)$, *then there exists* $v \in M_{(13)(23)}[e] - 0$ *such that* $W_{\mathbf{t}}(\Bbbk_e, L) \simeq \mathcal{A}_{[\mathbf{a}]} \cdot v$.

(c) *If* $v \in M_{(13)(23)}[e] - 0$, *then there exists* $\mathbf{t} \neq (0,0,0)$ *such that* $W_{\mathbf{t}}(\Bbbk_e, L) \simeq \mathcal{A}_{[\mathbf{a}]} \cdot v$.

(d) $W_{\mathbf{t}}(\Bbbk_e, L)$ *is an extension of* $\Bbbk_e$ *by* L.

(e) $W_{\mathbf{t}}(\Bbbk_e, L) \simeq W_{\tilde{\mathbf{t}}}(\Bbbk_e, L)$ *if and only if* $\mathbf{t} = \mu\tilde{\mathbf{t}}$ *with* $\mu \in \Bbbk^\times$.

Proof. (a) is immediate. If we prove (b), then (d) follows from Lemma 13.

(b) We set $w_{(13)(23)} = m_{\mathsf{soc}} \in M_{(13)(23)}[(13)(23)]$,

$$w_{(23)} = \frac{x_{(13)} \cdot w_{(13)(23)}}{f_{13}((13)(23))}, \; w_{(13)} = \frac{x_{(12)} \cdot w_{(13)(23)}}{f_{12}((13)(23))}, \quad w_{(12)} = \frac{x_{(23)} \cdot w_{(13)(23)}}{f_{23}((13)(23))},$$

$w_{(23)(13)} = x_{(23)}x_{(12)} \cdot w_{(13)(23)}$ and $w_e = -t_{(12)} m_{(13)(12)} + t_{(13)} m_{(12)(23)} \neq 0$ Using the formulae (23) to (49), it is not difficult to see that $W_{\mathbf{t}}(\Bbbk_e, L) \simeq \mathcal{A}_{[\mathbf{a}]} \cdot w_e$. (c) follows using the formulae (23) to (49). The proof of (e) is similar to the proof of Lemma 12 (e). $\square$

Theorem 1. *Let $\mathbf{a} \in \mathfrak{A}_3$ be generic. There are exactly 2 simple $\mathcal{A}_{[\mathbf{a}]}$modules up to isomorphism, namely $\Bbbk_e$ and L. Moreover, M_e is the projective cover, and the injective hull, of $\Bbbk_e$; also, $M_{(13)(23)}$ is the projective cover, and the injective hull, of L.*

Proof. We know that $\Bbbk_e$ and L are the only two simple $\mathcal{A}_{[\mathbf{a}]}$-modules up to isomorphism by Proposition 1 and Lemmata 7 (a), 10 and 13. Hence, a set of primitive orthogonal idempotents has at most 6 elements [CR, (6.8)]. Since the δ_g, $g \in \mathbb{S}_3$ are orthogonal idempotents, they must be primitive. Therefore M_e and $M_{(13)(23)}$ are the projective covers (and the injective hulls) of $\Bbbk_e$ and L, respectively by [CR, (9.9)], see page 3. $\square$

3.3. **Case $\mathbf{a} \in \mathfrak{A}_3$ sub-generic.**

Through this subsection, we suppose that $a_{(12)} \neq a_{(13)} = a_{(23)}$. Then the equivalence classes of $\mathbb{S}_3$ by $\sim_{\mathbf{a}}$ are

$$\{e\}, \qquad \{(12)\} \qquad \text{and } \{(13), (23), (13)(23), (23)(13)\}.$$

In fact,

- e and (12) belong to the isotropy group $\mathbb{S}_3^{\mathbf{a}}$.
- $(13) = (23)(12)(23)$ with $f_{12}((23)) = a_{(12)} - a_{(13)} \neq 0$ and $f_{23}((12)(23)) = a_{(23)} - a_{(12)} \neq 0$.
- $(123) = (13)(23)$ with $f_{13}((23)) = a_{(13)} - a_{(12)} \neq 0$.
- $(132) = (23)(13)$ with $f_{23}((13)) = a_{(23)} - a_{(12)} \neq 0$.

To determine the simple $\mathcal{A}_{[\mathbf{a}]}$-modules, we proceed as in the subsection above; that is, we just need to determine the maximal submodules of the Verma modules M_e, $M_{(12)}$ and $M_{(13)(23)}$, see Proposition 5.

Let M be a cyclic $\mathcal{A}_{[\mathbf{a}]}$-module generated by $v \in M[(13)(23)]$. Here again, we can describe the weight spaces of M. By (55) and acting by the monomials in our basis, we see that $M[(23)(13)] = \langle x_{(13)}x_{(23)} \cdot v, x_{(23)}x_{(12)} \cdot v, x_{(12)}x_{(13)} \cdot v\rangle$. This weight space is $\neq 0$ by Remark 4 applied to $(13)(23) \sim_{\mathbf{a}} (23)(13)$, and a further application of this Remark gives the following result.

Remark 16. Let M be a cyclic $\mathcal{A}_{[\mathbf{a}]}$-module generated by $v \in M[(13)(23)]$. If $\dim M[(23)(13)] = 1$, then

$$\begin{aligned} M[e] &= \langle x_{(23)}x_{(13)} \cdot v, (x_{(12)}x_{(23)}) \cdot v, x_{(13)}x_{(12)} \cdot v\rangle, & M[(13)(23)] &= \langle v\rangle, \\ M[(12)] &= \langle x_{(23)} \cdot v, (x_{(13)}x_{(12)}x_{(13)}) \cdot v\rangle, & M[(23)] &= \langle x_{(13)} \cdot v\rangle, \\ M[(23)(13)] &= \langle x_{(12)}x_{(13)} \cdot v\rangle, & M[(13)] &= \langle x_{(12)} \cdot v\rangle. \end{aligned} \tag{60}$$

There is a simple module L like this; $\{v_{(13)}, v_{(23)}, v_{(13)(23)}, v_{(23)(13)}\}$ is a basis of L and the action is given by

$$v_g \in L[g], \qquad x_{(ij)} \cdot v_g = \begin{cases} 0 & \text{if } g = (ij) \\ m_{(ij)g} & \text{if } g \neq (ij), \operatorname{sgn} g = -1, \\ f_{ij}(g) m_{(ij)g} & \text{if } \operatorname{sgn} g = 1. \end{cases} \tag{61}$$

Let $\Bbbk_{(12)}$ and $\Bbbk_e$ be as in Lemma 8. We shall see that L, $\Bbbk_{(12)}$ and $\Bbbk_e$ are the only simple modules of $\mathcal{A}_{[\mathbf{a}]}$.

The Verma module M_e projects onto the simple module $\Bbbk_e$, hence the kernel of this projection is a maximal submodule; explicitly this is

$$N_e = \mathcal{A}_{[\mathbf{a}]} \cdot (M_e[(13)(23)] \oplus M_e[(12)]) = \oplus_{g \sim_{\mathbf{a}} (13)(23)} M_e[g] \oplus M_e[(12)] \oplus \langle m_{\mathsf{top}} \rangle.$$

We see that this is the unique maximal submodule, as consequence of the following description of all submodules of M_e.

Lemma 17. *The lattice of (proper, non-trivial) submodules of M_e is displayed in* (62)*, where v and w satisfy*

$$M_e[(13)(23)] = \langle v, m_{(23)(12)} \rangle, \qquad M_e[(12)] = \langle w, m_{(13)(12)(23)} \rangle.$$

The submodules $\mathcal{A}_{[\mathbf{a}]} \cdot v$ (resp. $\mathcal{A}_{[\mathbf{a}]} \cdot w$) and $\mathcal{A}_{[\mathbf{a}]} \cdot v_1$ (resp. $\mathcal{A}_{[\mathbf{a}]} \cdot w_1$) coincide iff $v \in \langle v_1 \rangle$ (resp. $w \in \langle w_1 \rangle$). The labels on the arrows indicate the quotient of the module on top by the module on the bottom.

(62)
$$\begin{tikzcd}
& & N_e \arrow[dll, dash, "\Bbbk_{(12)}"'] \arrow[drr, dash, "L"] & & \\
\mathcal{A}_{[\mathbf{a}]} \cdot M_e[(13)(23)] \arrow[d, dash, "L"'] \arrow[drr, dash, "L"] & & & & \mathcal{A}_{[\mathbf{a}]} \cdot M_e[(12)] \arrow[dll, dash, "\Bbbk_{(12)}"'] \arrow[d, dash, "\Bbbk_{(12)}"] \\
\mathcal{A}_{[\mathbf{a}]} \cdot v \arrow[d, dash, "L"'] & & \mathcal{A}_{[\mathbf{a}]} \cdot \langle m_{(13)(12)(23)}, m_{(23)(12)} \rangle \arrow[dll, dash, "L"'] \arrow[drr, dash, "\Bbbk_{(12)}"] & & \mathcal{A}_{[\mathbf{a}]} \cdot w \arrow[d, dash, "\Bbbk_{(12)}"] \\
\mathcal{A}_{[\mathbf{a}]} \cdot m_{(13)(12)(23)} \arrow[drr, dash, "\Bbbk_{(12)}"'] & & & & \mathcal{A}_{[\mathbf{a}]} \cdot m_{(23)(12)} \arrow[dll, dash, "L"] \\
& & \langle m_{\mathsf{top}} \rangle & &
\end{tikzcd}$$

Proof. Let

$$\begin{aligned} v &= \lambda m_{(23)(12)} + \mu m_{(12)(13)} && \in M_e[(13)(23)] - 0, \\ \tilde{v} &= \mu m_{(12)(23)} + (\mu - \lambda) m_{(13)(12)} && \in M_e[(23)(13)]. \end{aligned}$$

Using the formulae (23) to (49), we see that $x_{(23)}x_{(12)} \cdot v$ and $x_{(12)}x_{(13)} \cdot v$ are non-zero multiples of $\tilde{v}$. That is, $\dim(\mathcal{A}_{[\mathbf{a}]} \cdot v)[(23)(13)] = 1$. Moreover, $x_{(12)}x_{(23)} \cdot v = -\mu m_{\mathsf{top}}$ and $x_{(13)}x_{(12)} \cdot v = \lambda m_{\mathsf{top}}$; and $x_{(23)} \cdot v$ and $\big(x_{(13)}x_{(12)}x_{(13)}\big) \cdot v$ are non-zero multiples of $\mu m_{(13)(12)(23)}$. By Remark 16, we obtain a basis for $\mathcal{A}_{[\mathbf{a}]} \cdot v$:

$$\left\{ v,\, x_{(12)} \cdot v,\, x_{(13)} \cdot v,\, \tilde{v},\, m_{\mathsf{top}},\, \mu m_{(13)(12)(23)} \right\}; \tag{63}$$

if $\mu = 0$, we obviate the last vector.

By (51), (50) and (52), $x_{(ij)} \cdot m_{\mathsf{top}} = 0$ for all $(ij) \in \mathcal{O}_2^3$. Then

$$\mathcal{A}_{[\mathbf{a}]} \cdot m_{\mathsf{top}} = \langle m_{\mathsf{top}} \rangle$$

and $\mathcal{A}_{[\mathbf{a}]} \cdot u = \mathcal{A}_{[\mathbf{a}]} \cdot m_1 = M_e$ if $u \in M_e[e]$ is linearly independent to m_{top}.

By (43), (46) and (49), $x_{(ij)} \cdot m_{(13)(12)(23)} = -\delta_{(12)}((\mathrm{ij}))m_{\mathsf{top}}$ for all $(ij) \in \mathcal{O}_2^3$. Then

$$\mathcal{A}_{[\mathbf{a}]} \cdot m_{(13)(12)(23)} = \langle m_{\mathsf{top}}, m_{(13)(12)(23)} \rangle.$$

By (22), (24) and (26), $x_{(ij)} \cdot m_{(12)} = \delta_{(13)}((\mathrm{ij}))m_{(13)(12)} + \delta_{(23)}((\mathrm{ij}))m_{(23)(12)}$ for all $(ij) \in \mathcal{O}_2^3$. Then

$$\mathcal{A}_{[\mathbf{a}]} \cdot w = \mathcal{A}_{[\mathbf{a}]} \cdot m_{(23)(12)} \oplus \langle w \rangle$$

by (63) and Remark 4, if $w \in M_e[(12)]$ is linearly independent to $m_{(13)(12)(23)}$.

Let now N be a (proper, non-trivial) submodule of M_e which is not $\langle m_{\mathsf{top}} \rangle$. We set $\widetilde{N} = \mathcal{A}_{[\mathbf{a}]} \cdot N[(12)] + \mathcal{A}_{[\mathbf{a}]} \cdot N[(13)(23)]$. Then $\widetilde{N}[g] = N[g]$ for all $g \neq e$ by Remark 4. By the argument at the beginning of the proof, $\langle m_{\mathsf{top}} \rangle \subset \widetilde{N}$. Then $\widetilde{N}[e] = \langle m_{\mathsf{top}} \rangle = N[e]$ because otherwise $N = M_e$. Therefore $N = \widetilde{N}$. To finish, we have to calculate the submodules of M_e generated by homogeneous subspaces of $M_e[(12)] \oplus M_e[(13)(23)]$; this follows from the argument at the beginning of the proof. $\square$

The Verma module $M_{(13)(23)}$ projects onto the simple module L, hence the kernel of this projection is a maximal submodule; explicitly this is

$$\begin{aligned} N_{(13)(23)} &= \mathcal{A}_{[\mathbf{a}]} \cdot \big(M_{(13)(23)}[e] \oplus M_{(13)(23)}[(12)]\big) \\ &= M_{(13)(23)}[e] \oplus M_{(13)(23)}[(12)] \oplus \mathcal{A}_{[\mathbf{a}]} \cdot m_{\mathsf{soc}}. \end{aligned}$$

We see that this is the unique maximal submodule, as consequence of the following description of all submodules of $M_{(13)(23)}$.

Lemma 18. *The lattice of (proper, non-trivial) submodules of $M_{(13)(23)}$ is*

$$\begin{tikzcd}[arrows=dash]
 & & N_{(13)(23)} \arrow[dll, "\Bbbk_{(12)}"'] \arrow[drr, "\Bbbk_e"] & & \\
\mathcal{A}_{[\mathbf{a}]} \cdot M_{(13)(23)}[e] \arrow[d, "\Bbbk_e"'] \arrow[drr, "\Bbbk_e"] & & & & \mathcal{A}_{[\mathbf{a}]} \cdot M_{(13)(23)}[(12)] \arrow[dll, "\Bbbk_{(12)}"'] \arrow[d, "\Bbbk_{(12)}"] \\
\mathcal{A}_{[\mathbf{a}]} \cdot v \arrow[d, "\Bbbk_e"'] & & \mathcal{A}_{[\mathbf{a}]} \cdot \langle m_{\mathsf{o}}, m_{(12)(23)} \rangle \arrow[dll, "\Bbbk_e"] \arrow[drr, "\Bbbk_{(12)}"'] & & \mathcal{A}_{[\mathbf{a}]} \cdot w \arrow[d, "\Bbbk_{(12)}"] \\
\mathcal{A}_{[\mathbf{a}]} \cdot m_{\mathsf{o}} \arrow[drr, "\Bbbk_{(12)}"'] & & & & \mathcal{A}_{[\mathbf{a}]} \cdot m_{(12)(23)} \arrow[dll, "\Bbbk_e"] \\
 & & \mathcal{A}_{[\mathbf{a}]} \cdot m_{\mathsf{soc}} & &
\end{tikzcd}$$

Here v and w satisfy $M_{(13)(23)}[e] = \langle v, m_{(12)(23)} \rangle$, $M_{(13)(23)}[(12)] = \langle w, m_{\mathsf{o}} \rangle$. The submodules $\mathcal{A}_{[\mathbf{a}]} \cdot v$ (resp. $\mathcal{A}_{[\mathbf{a}]} \cdot w$) and $\mathcal{A}_{[\mathbf{a}]} \cdot v_1$ (resp. $\mathcal{A}_{[\mathbf{a}]} \cdot w_1$) coincide iff $v \in \langle v_1 \rangle$ (resp. $w \in \langle w_1 \rangle$). The labels on the arrows indicate the quotient of the module on top by the module on the bottom.

Proof. Let $u = \lambda m_1 + \mu m_{\mathsf{top}} \in M_{(13)(23)}[(13)(23)] - 0$. Using the formulae (23) to (49), we see that

$$x_{(12)}x_{(13)} \cdot u = \lambda m_{(12)(13)} - \mu f_{13}((23))^2 m_{(23)(12)} \text{ and}$$
$$x_{(23)}x_{(12)} \cdot u = \mu f_{23}((13))^2 m_{(12)(13)} + \big(\lambda + 2\mu f_{13}((23)) f_{23}((13))\big) m_{(23)(12)}.$$

Thus, $\dim N[(23)(13)] = 1$ iff $\lambda + \mu f_{13}((23)) f_{23}((13)) = 0$, that is iff $u \in \langle m_{\mathsf{soc}} \rangle - 0$. By Remark 16,

$$\mathcal{A}_{[\mathbf{a}]} \cdot m_{\mathsf{soc}} = \langle m_{\mathsf{soc}},\, x_{(12)} \cdot m_{\mathsf{soc}},\, x_{(13)} \cdot m_{\mathsf{soc}},\, x_{(12)}x_{(13)} \cdot m_{\mathsf{soc}} \rangle$$

and $\mathcal{A}_{[\mathbf{a}]} \cdot u = \mathcal{A}_{[\mathbf{a}]} \cdot m_1 = M_{(13)(23)}$, if $u \in M_{(13)(23)}[(13)(23)]$ is linearly independent to m_{soc}.

By the formulae (23) to (52), if $u \in \big(M_{(13)(23)}[e] \oplus M_{(13)(23)}[(12)]\big) - 0$, then $0 \neq \langle x_{(13)} \cdot u, x_{(23)} \cdot u \rangle \subset \mathcal{A}_{[\mathbf{a}]} \cdot m_{\mathsf{soc}}$. Therefore

$$\mathcal{A}_{[\mathbf{a}]} \cdot m_{\mathsf{soc}} \subset \mathcal{A}_{[\mathbf{a}]} \cdot u$$

by Remark 4. Also, if v and w satisfy $M_{(13)(23)}[e] = \langle v, m_{(12)(23)} \rangle$ and $M_{(13)(23)}[(12)] = \langle w, m_{\mathsf{o}} \rangle$, then

$$\langle x_{(12)} \cdot v \rangle = \langle m_{\mathsf{o}} \rangle \quad \text{and} \quad \langle x_{(12)} \cdot w \rangle = \langle m_{(12)(23)} \rangle.$$

Let now N be a (proper, non-trivial) submodule of $M_{(13)(23)}$ which is not $\mathcal{A}_{[\mathbf{a}]} \cdot m_{\mathsf{soc}}$. We set $\widetilde{N} = \mathcal{A}_{[\mathbf{a}]} \cdot N[e] + \mathcal{A}_{[\mathbf{a}]} \cdot N[(12)]$. Then $\widetilde{N}[g] = N[g]$ for $g = e, (12)$ by Remark 4. By the argument at the beginning of the proof, $\mathcal{A}_{[\mathbf{a}]} \cdot m_{\mathsf{soc}} \subset \widetilde{N}$. Then $\oplus_{g \sim_{\mathbf{a}} (13)(23)} N[g] = \mathcal{A}_{[\mathbf{a}]} \cdot m_{\mathsf{soc}} = \oplus_{g \sim_{\mathbf{a}} (13)(23)} \widetilde{N}[g]$ because otherwise $N = M_{(13)(23)}$. Therefore $N = \widetilde{N}$. To finish, we have to calculate the submodules of $M_{(13)(23)}$ generated by homogeneous subspaces of $M_{(13)(23)}[(12)] \oplus M_{(13)(23)}[e]$; this follows from the argument at the beginning of the proof. □

The Verma module $M_{(12)}$ projects onto the simple module $\Bbbk_{(12)}$, hence the kernel of this projection is a maximal submodule; explicitly this is

$$\begin{aligned} N_{(12)} &= \mathcal{A}_{[\mathbf{a}]} \cdot \big(M_{(12)}[(13)(23)] \oplus M_{(12)}[e]\big) \\ &= \oplus_{g \sim_{\mathbf{a}} (13)(23)} M_{(12)}[g] \oplus M_{(12)}[e] \oplus \langle m_{\mathsf{top}} \rangle. \end{aligned}$$

We see that this is the unique maximal submodule, as consequence of the following description of all submodules of $M_{(12)}$.

Lemma 19. *The lattice of (proper, non-trivial) submodules of* $M_{(12)}$ *is*

$$\begin{tikzcd}
 & & N_{(12)} \arrow[dll, dash, "\Bbbk_e"'] \arrow[drr, dash, "L"] & & \\
\mathcal{A}_{[\mathbf{a}]}\cdot M_{(12)}[(13)(23)] \arrow[d, dash, "L"'] \arrow[drr, dash, "L"] & & & & \mathcal{A}_{[\mathbf{a}]}\cdot M_{(12)}[e] \arrow[dll, dash, "\Bbbk_e"'] \arrow[d, dash, "\Bbbk_e"] \\
\mathcal{A}_{[\mathbf{a}]}\cdot v \arrow[d, dash, "L"'] & & \mathcal{A}_{[\mathbf{a}]}\cdot \langle m_{(13)(12)(23)}, m_{\mathsf{o}}\rangle \arrow[dll, dash, "L"] \arrow[drr, dash, "\Bbbk_e"] & & \mathcal{A}_{[\mathbf{a}]}\cdot w \arrow[d, dash, "\Bbbk_e"] \\
\mathcal{A}_{[\mathbf{a}]}\cdot m_{(13)(12)(23)} \arrow[drr, dash, "\Bbbk_e"'] & & & & \mathcal{A}_{[\mathbf{a}]}\cdot m_{\mathsf{o}} \arrow[dll, dash, "L"] \\
 & & \langle m_{\mathsf{top}}\rangle & &
\end{tikzcd}$$

Here v *and* w *satisfy* $M_{(12)}[(13)(23)] = \langle v, m_{\mathsf{o}}\rangle$, $M_{(12)}[e] = \langle w, m_{(13)(12)(23)}\rangle$. *The submodules* $\mathcal{A}_{[\mathbf{a}]}\cdot v$ *(resp.* $\mathcal{A}_{[\mathbf{a}]}\cdot w$*) and* $\mathcal{A}_{[\mathbf{a}]}\cdot v_1$ *(resp.* $\mathcal{A}_{[\mathbf{a}]}\cdot w_1$*) coincide iff* $v \in \langle v_1\rangle$ *(resp.* $w \in \langle w_1\rangle$*). The labels on the arrows indicate the quotient of the module on top by the module on the bottom.*

Proof. Let $v = \lambda m_{(23)} + \mu m_{(13)(12)(13)} \in M_{(12)}[(13)(23)]$ be a non-zero element. By Remark 16 and using the formulae (23) to (52), we see that

$$\begin{aligned}
(\mathcal{A}_{[\mathbf{a}]}\cdot v)[(13)(23)] &= \langle v\rangle,\\
(\mathcal{A}_{[\mathbf{a}]}\cdot v)[(13)] &= \langle (f_{13}((23))\mu - \lambda)m_{(12)(23)} - \mu f_{13}((23))m_{(13)(12)}\rangle,\\
(\mathcal{A}_{[\mathbf{a}]}\cdot v)[(23)] &= \langle (f_{13}((23))\mu - \lambda)m_{(12)(13)} - \lambda m_{(23)(12)}\rangle,\\
(\mathcal{A}_{[\mathbf{a}]}\cdot v)[(23)(13)] &= \langle (f_{13}((23))\mu - \lambda)f_{23}((13))m_{(13)} + \lambda m_{(12)(23)(12)}\rangle,\\
(\mathcal{A}_{[\mathbf{a}]}\cdot v)[(12)] &= \langle m_{\mathsf{top}}\rangle \text{ and}\\
(\mathcal{A}_{[\mathbf{a}]}\cdot v)[e] &= \langle (f_{13}((23))\mu - \lambda)m_{(13)(12)(23)}\rangle.
\end{aligned} \tag{64}$$

By (51), (50) and (52), $x_{(ij)}\cdot m_{\mathsf{top}} = 0$ for all $(ij) \in \mathcal{O}_2^3$. Then

$$\mathcal{A}_{[\mathbf{a}]}\cdot m_{\mathsf{top}} = \langle m_{\mathsf{top}}\rangle$$

and $\mathcal{A}_{[\mathbf{a}]}\cdot u = \mathcal{A}_{[\mathbf{a}]}\cdot m_1 = M_e$, if $u \in M_{(12)}[(12)]$ is linearly independent to m_{top}. By (43), (46) and (49), $x_{(ij)}\cdot m_{(13)(12)(23)} = -\delta_{(12)}((\mathrm{ij}))m_{\mathsf{top}}$ for all $(ij) \in \mathcal{O}_2^3$. Then

$$\mathcal{A}_{[\mathbf{a}]}\cdot m_{(13)(12)(23)} = \langle m_{\mathsf{top}}, m_{(13)(12)(23)}\rangle.$$

By (22), (24) and (26), $x_{(ij)}\cdot m_{(12)} = \delta_{(13)}((\mathrm{ij}))m_{(13)(12)} + \delta_{(23)}((\mathrm{ij}))m_{(23)(12)}$ for all $(ij) \in \mathcal{O}_2^3$. Then

$$\mathcal{A}_{[\mathbf{a}]}\cdot w = \mathcal{A}_{[\mathbf{a}]}\cdot m_{\mathsf{o}} \oplus \langle w\rangle$$

by (64) and Remark 4, if $w \in M_{(12)}[e]$ is linearly independent to $m_{(13)(12)(23)}$.

Let now N be a (proper, non-trivial) submodule of $M_{(12)}$ which is not $\langle m_{\mathsf{top}}\rangle$. We set $\widetilde{N} = \mathcal{A}_{[\mathbf{a}]} \cdot N[e] + \mathcal{A}_{[\mathbf{a}]} \cdot N[(13)(23)]$. Then $\widetilde{N}[g] = N[g]$ for all $g \neq (12)$ by Remark 4. By the argument at the beginning of the proof, $\langle m_{\mathsf{top}}\rangle \subset \widetilde{N}$. Then $N[(12)] = \langle m_{\mathsf{top}}\rangle = \tilde{N}[(12)]$ because otherwise $N = M_{(12)}$. Therefore $N = \widetilde{N}$. To finish, we have to calculate the submodules of $M_{(12)}$ generated by homogeneous subspaces of $M_{(12)}[(13)(23)] \oplus M_{(12)}[e]$; this follows from the argument at the beginning of the proof. □

As a consequence, we obtain the simples modules in the sub-generic case. The proof of the next theorem runs in the same way as that of Theorem 1.

Theorem 2. *Let $\mathbf{a} \in \mathfrak{A}_3$ with $a_{(12)} \neq a_{(13)} = a_{(23)}$. There are exactly 3 simple $\mathcal{A}_{[\mathbf{a}]}$modules up to isomorphism, namely $\Bbbk_e$, $\Bbbk_{(12)}$ and L. Moreover, M_e is the projective cover, and the injective hull, of $\Bbbk_e$; $M_{(12)}$ is the projective cover, and the injective hull, of $\Bbbk_{(12)}$; and $M_{(13)(23)}$ is the projective cover, and the injective hull, of L.*

Proof. We know that $\Bbbk_e$, $\Bbbk_{(12)}$ and L are the only two simple $\mathcal{A}_{[\mathbf{a}]}$-modules up to isomorphism by Proposition 1 and Lemmata 17, 18 and 19. Hence, a set of primitive orthogonal idempotents has at most 6 elements [CR, (6.8)]. Since the δ_g, $g \in \mathbb{S}_3$ are orthogonal idempotents, they must be primitive. Therefore M_e, $M_{(12)}$ and $M_{(13)(23)}$ are respectively the projective covers (and the injective hulls) of $\Bbbk_e$, $\Bbbk_{(12)}$ and L by [CR, (9.9)], see page 3. □

4. Representation type of $\mathcal{A}_{[\mathbf{a}]}$

In this section, we assume that $n = 3$ as in the preceding one. We will determine the $\mathcal{A}_{[\mathbf{a}]}$-modules which are extensions of simple $\mathcal{A}_{[\mathbf{a}]}$-modules. As a consequence, we will show that $\mathcal{A}_{[\mathbf{a}]}$ is not of finite representation type for all $\mathbf{a} \in \mathfrak{A}_3$.

4.1. **Extensions of simple modules.** By the following lemma, we are reduced to consider only submodules of the Verma modules for to determine the extensions of simple $\mathcal{A}_{[\mathbf{a}]}$-modules. Then we shall split the consideration into three different cases like Section 3 and use the lemmata there.

Lemma 20. *Let $\mathbf{a} \in \mathfrak{A}_3$ be non-zero. Let S and T be simple $\mathcal{A}_{[\mathbf{a}]}$-modules and M be an extension of T by S. Hence either $M \simeq S \oplus T$ as $\mathcal{A}_{[\mathbf{a}]}$-modules or M is an indecomposable submodule of the Verma module which is the injective hull of S.*

Proof. If there exists a proper submodule N of M which is not S, then $M \simeq S \oplus T$ as $\mathcal{A}_{[\mathbf{a}]}$-modules. In fact, $N \cap S$ is either 0 or S because S is simple. Let π be as in (65). Since T is simple, $\pi_{|N} : N \to T$ results an epimorphism. Therefore $M \simeq S \oplus T$ since $\dim N = \dim(N \cap S) + \dim T$.

Let M_S be the Verma module which is the injective hull of S. Then we have the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & S \arrow[r, "\imath"] \arrow[d, hook] & M \arrow[r, "\pi"] \arrow[dl, dashed, "f"] & T \arrow[r] & 0 \\ & M_S & & & \end{tikzcd} \tag{65}$$

Therefore either $M \simeq S \oplus T$ as $\mathcal{A}_{[\mathbf{a}]}$-modules or f is inyective. If f is inyective, then M results indecomposable by Lemmata 10 and 13 in the generic case, and by Lemmata 17, 18 and 19 in the sub-generic case. $\square$

Recall the modules $W_{\mathbf{t}}(L, \Bbbk_e)$ and $W_{\mathbf{t}}(\Bbbk_e, L)$ from Definitions 11 and 14. The next results follow from Lemmata 10, 13, 17, 18 and 19 by Lemma 20.

Lemma 21. *Let $\mathbf{a} \in \mathfrak{A}_3$ be generic. Let S and T be simple $\mathcal{A}_{[\mathbf{a}]}$-modules and M be an extension of T by S.*

(a) *If $S \simeq T$, then $M \simeq S \oplus S$.*
(b) *If $S \simeq \Bbbk_e$ and $T \simeq L$, then $M \simeq W_{\mathbf{t}}(L, \Bbbk_e)$ for some $\mathbf{t} \in \mathfrak{A}_3$.*
(c) *If $S \simeq L$ and $T \simeq \Bbbk_e$, then $M \simeq W_{\mathbf{t}}(\Bbbk_e, L)$ for some $\mathbf{t} \in \mathfrak{A}_3$.* $\square$

Lemma 22. *Let $\mathbf{a} \in \mathfrak{A}_3$ with $a_{(12)} \neq a_{(13)} = a_{(23)}$. Let S and T be simple $\mathcal{A}_{[\mathbf{a}]}$-modules and M be an extension of T by S.*

(a) *If $S \simeq T$, then $M \simeq S \oplus S$.*
(b) *If $S \simeq \Bbbk_e$ and $T \simeq \Bbbk_{(12)}$, then $M \simeq \mathcal{A}_{[\mathbf{a}]} \cdot m_{(13)(12)(23)} \subset M_e$.*
(c) *If $S \simeq \Bbbk_{(12)}$ and $T \simeq \Bbbk_e$, then $M \simeq \mathcal{A}_{[\mathbf{a}]} \cdot m_{(13)(12)(23)} \subset M_{(12)}$.*
(d) *If $S \simeq \Bbbk_e$ and $T \simeq L$, then $M \simeq \mathcal{A}_{[\mathbf{a}]} \cdot m_{(23)(12)} \subset M_e$.*
(e) *If $S \simeq L$ and $T \simeq \Bbbk_e$, then $M \simeq \mathcal{A}_{[\mathbf{a}]} \cdot m_{(12)(23)} \subset M_{(13)(23)}$.*
(f) *If $S \simeq \Bbbk_{(12)}$ and $T \simeq L$, then $M \simeq \mathcal{A}_{[\mathbf{a}]} \cdot m_{\mathsf{o}} \subset M_{(12)}$.*
(g) *If $S \simeq L$ and $T \simeq \Bbbk_{(12)}$, then $M \simeq \mathcal{A}_{[\mathbf{a}]} \cdot m_{\mathsf{o}} \subset M_{(13)(23)}$.* $\square$

Lemma 23. *Let $\Bbbk_g$ and $\Bbbk_h$ be one-dimensional simple $\mathcal{A}_{[(0,0,0)]}$-modules and M be an extension of $\Bbbk_h$ by $\Bbbk_g$. Hence*

(a) *If $\operatorname{sgn} g = \operatorname{sgn} h$, then $M \simeq \Bbbk_g \oplus \Bbbk_h$.*
(b) *If $\operatorname{sgn} g \neq \operatorname{sgn} h$ and M is not isomorphic to $\Bbbk_g \oplus \Bbbk_h$, then $g = (st)h$ for a unique $(st) \in \mathcal{O}_3^2$ and M has a basis $\{w_g, w_h\}$ such that $\langle w_g \rangle \simeq \Bbbk_g$ as $\mathcal{A}_{[\mathbf{a}]}$-modules, $w_h \in M[h]$ and $x_{(ij)} w_h = \delta_{(ij),(st)} w_g$.*

Proof. $M = M[g] \oplus M[h]$ as $\Bbbk^{\mathbb{S}_3}$-modules and $M[g] \simeq \Bbbk_g$ as $\mathcal{A}_{[\mathbf{a}]}$-modules. Since $x_{(ij)} \cdot M[h] \subset M[(\mathrm{ij})h]$, the lemma follows. $\square$

4.2. **Representation type.** We summarize some facts about the representation type of an algebra.

Let R be an algebra and $\{S_1, ..., S_t\}$ be a complete list of non-isomorphic simple R-modules. The *separated quiver of* R is constructed as follows. The set of vertices is $\{S_1, ..., S_t, S_1', ..., S_t'\}$ and we write $\dim \operatorname{Ext}^1_R(S_i, S_j)$ arrows from S_i to S_j', cf. [ARS, p. 350]. Let us denote by Γ_R the underlying graph of the separated quiver of R.

A characterization of the hereditary algebras of finite and tame representation type is well-known, see for example [DR2]. As a consequence, the next well-known result is obtained. If R is of finite representation type, then it is Theorem D of [DR1] or Theorem X.2.6 of [ARS]. The proof given in [ARS] adapts immediately to the case when R is of tame representation type.

Theorem 3. *Let R be a finite dimensional algebra with radical square zero. Then R is of finite (resp. tame) representation type if and only if Γ_R is a finite (resp. affine) disjoint union of Dynkin diagrams.* $\square$

In order to use the above theorem, we know that

Remark 24. If $\mathfrak{r}$ is the radical of R, then the separated quiver of R is equal to the separated quiver of $R/\mathfrak{r}^2$, see for example [GI, Lemma 4.5].

We obtain the following result by combining Corollary VI.1.5 and Proposition VI.1.6 of [ARS].

Proposition 25. *Let R be an artin algebra, χ an infinite cardinal and assume there are χ non-isomorphic indecomposable modules of length n. Then R is not of finite representation type.* $\square$

Here is the announced result.

Proposition 26. *$\mathcal{A}_{[(0,0,0)]}$ is of wild representation type. If $\mathbf{a} \in \mathfrak{A}_3$ is non-zero, then $\mathcal{A}_{[\mathbf{a}]}$ is not of finite representation type.*

Proof. If $\mathbf{a} \in \mathfrak{A}_3$ is generic, we can apply Proposition 25 by Lemma 12 and Lemma 15. Hence $\mathcal{A}_{[\mathbf{a}]}$ is not of finite representation type for all $\mathbf{a} \in \mathfrak{A}_3$ generic.

Let $\mathbf{a} \in \mathfrak{A}_3$ be sub-generic or zero. Then $\dim \operatorname{Ext}^1_{\mathcal{A}_{[\mathbf{a}]}}(T,S) = 0$ if $S \simeq T$ by Lemma 22 and 23, and $\dim \operatorname{Ext}^1_{\mathcal{A}_{[\mathbf{a}]}}(T,S) = 1$ in otherwise. In fact, suppose that $a_{(12)} \neq a_{(13)} = a_{(23)}$, $S \simeq \Bbbk_e$ and $T \simeq L$. By Lemma 18 and Therorem 2, L admits a projective resolution of the form

$$\dots \longrightarrow P^2 \longrightarrow M_e \oplus M_{(12)} \xrightarrow{F} M_{(13)(23)} \longrightarrow L \longrightarrow 0,$$

where F is defined by $F_{|M_e}(m_1) = v$ and $F_{|M_{(12)}}(m_1) = w$; here v and w satisfy $M_{(13)(23)}[e] = \langle v, m_{(12)(23)} \rangle$, $M_{(13)(23)}[(12)] = \langle w, m_{\mathsf{o}} \rangle$. Then

$$0 \longrightarrow \operatorname{Hom}_{\mathcal{A}_{[\mathbf{a}]}}(M_{(13)(23)}, \Bbbk_e) \xrightarrow{\partial_0} \operatorname{Hom}_{\mathcal{A}_{[\mathbf{a}]}}(M_e \oplus M_{(12)}, \Bbbk_e) \xrightarrow{\partial_1} \dots$$

and $\operatorname{Ext}^1_{\mathcal{A}_{[\mathbf{a}]}}(L, \Bbbk_e) = \ker \partial_1 / \operatorname{Im} \partial_0$. Since M_h is generated by $m_1 \in M_h[h]$ for all $h \in \mathbb{S}_3$, $\operatorname{Hom}_{\mathcal{A}_{[\mathbf{a}]}}(M_{(13)(23)}, \Bbbk_e) = 0$ and $\dim \operatorname{Hom}_{\mathcal{A}_{[\mathbf{a}]}}(M_e \oplus M_{(12)}, \Bbbk_e) = 1$. By Lemma 22, we know that there exists a non-trivial extension of L by $\Bbbk_e$ and therefore $\dim \operatorname{Ext}^1_{\mathcal{A}_{[\mathbf{a}]}}(L, \Bbbk_e) = 1$ because it is non-zero. For other S and T and for the case $\mathbf{a} = (0,0,0)$, the proof is similar.

Hence if $\mathbf{a} \in \mathcal{A}_{[\mathbf{a}]}$ is sub-generic and $a_{(12)} \neq a_{(13)} = a_{(23)}$, the separated quiver of $\mathcal{A}_{[\mathbf{a}]}$ is

$$\begin{tikzcd}
\Bbbk_e \arrow[r] \arrow[d] & \Bbbk'_{(12)} & L \arrow[l] \arrow[d] \\
L' & \Bbbk_{(12)} \arrow[l] \arrow[r] & \Bbbk'_e;
\end{tikzcd}$$

and the separated quiver of $\mathcal{A}_{[(0,0,0)]}$ is

$$\begin{tikzcd}
 & \Bbbk_e \arrow[dl] \arrow[d] \arrow[dr] & \\
\Bbbk'_{(12)} & \Bbbk'_{(13)} & \Bbbk'_{(23)} \\
\Bbbk_{(13)(23)} \arrow[u] \arrow[ur] \arrow[urr] & & \Bbbk_{(23)(13)} \arrow[u] \arrow[ul] \arrow[ull]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \Bbbk_{(12)} \arrow[dl] \arrow[d] \arrow[dr] & \\
\Bbbk'_{e} & \Bbbk'_{(13)(23)} & \Bbbk'_{(23)(13)} \\
\Bbbk_{(13)} \arrow[u] \arrow[ur] \arrow[urr] & & \Bbbk_{(23)} . \arrow[u] \arrow[ul] \arrow[ull]
\end{tikzcd}$$

Therefore the lemma follows from Theorem 3 and Remark 24. $\square$

Remark 27. Let $\mathbf{a} \in \mathfrak{A}_3$ be generic. It is not difficult to prove that the separated quiver of $\mathcal{A}_{[\mathbf{a}]}$ is

$$\Bbbk_e \rightrightarrows L' \qquad\qquad L \rightrightarrows \Bbbk'_e.$$

5. On the structure of $\mathcal{A}_{[\mathbf{a}]}$

In this section, we assume that $n = 3$ as in the preceding one.

5.1. **Cocycle deformations.**

We show in this subsection that the algebras $\mathcal{A}_{[\mathbf{a}]}$ are cocycle deformation of each other. For this, we first recall the following theorem due to Masuoka.

If K is a Hopf subalgebra of a Hopf algebra H and J is a Hopf ideal of K, then the two-sided ideal (J) of H is in fact a Hopf ideal of H.

Theorem 4. [M, Thm. 2], [BDR, Thm. 3.4]. *Suppose that K is Hopf subalgebra of a Hopf algebra H. Let I, J be Hopf ideal of K. If there is an algebra map ψ from K to $\Bbbk$ such that*

- $J = \psi \rightharpoonup I \leftharpoonup \psi^{-1}$ *and*
- $H/(\psi \rightharpoonup I)$ *is nonzero,*

then $H/(\psi \rightharpoonup I)$ is a $(H/(I), H/(J))$-biGalois object and so the quotient Hopf algebras $H/(I)$, $H/(J)$ are monoidally Morita-Takeuchi equivalent. If $H/(I)$ and $H/(J)$ are finite dimensional, then $H/(I)$ and $H/(J)$ are cocycle deformations of each other. $\square$

We will need the following lemma to apply the Masuoka's theorem.

Lemma 28. *If W is a vector space and U is a vector subspace of $W^{\otimes n}$, then the subalgebra of $T(W)$ generated by U is isomorphic to $T(U)$.*

Proof. It is enough to prove the lemma for $U = W^{\otimes n}$. Fix n and let $(x_i)_{i\in I}$ be a basis of W. Then $\mathbf{B} = \{X_{\mathbf{i}} = x_{i_1}\cdots x_{i_n} : \mathbf{i} = (i_1, ..., i_n) \in I^{\times n}\}$ forms a basis of $W^{\otimes n}$. Since the $X_{\mathbf{i}}$'s are all homogeneous elements of the same degree in $T(W)$, we only have to prove that $\{X_{\mathbf{i}_1}\cdots X_{\mathbf{i}_m} : \mathbf{i}_1, ..., \mathbf{i}_m \in I^{\times n}\}$ is linearly independent in $T(W)$ for all $m \geq 1$ and this is true because $\mathbf{B}$ is a basis of monomials of the same degree. □

Here is the announced result. Observe that this gives an alternative proof to the fact that $\dim \mathcal{A}_{[\mathbf{a}]} = 72$, proved in [AV] using the Diamond Lemma.

Proposition 29. *For all* $\mathbf{a} \in \mathfrak{A}_3$*,* $\mathcal{A}_{[\mathbf{a}]}$ *is a Hopf algebra monoidally Morita-Takeuchi equivalent to* $\mathcal{B}(V_3)\#\Bbbk^{\mathbb{S}_3}$*.*

Proof. To start with, we consider the algebra $\mathcal{K}_{\mathbf{a}} := T(V_3)\#\Bbbk^{\mathbb{S}_3}/\mathcal{J}_{\mathbf{a}}$, $\mathbf{a} \in \mathfrak{A}_3$, where $\mathcal{J}_{\mathbf{a}}$ is the ideal generated by

$$R_{(13)(23)}, \quad R_{(23)(13)} \quad \text{and} \quad x_{(ij)}^2 + \sum_{g\in\mathbb{S}_3} a_{g^{-1}(ij)g}\,\delta_g, \quad (ij) \in \mathcal{O}_2^3. \tag{66}$$

Let $M_3 = \Bbbk^{\mathbb{S}_3}$ with the regular representation. For all $\mathbf{a} \in \mathfrak{A}_3$, M_3 is an $\mathcal{K}_{\mathbf{a}}$-module with action given by

$$x_{(ij)} \cdot m_g = \begin{cases} m_{(ij)g} & \text{if } \operatorname{sgn} g = -1, \\ -a_{g^{-1}(ij)g}\, m_{(ij)g} & \text{if } \operatorname{sgn} g = 1. \end{cases}$$

We have to check that the relations defining $\mathcal{K}_{\mathbf{a}}$ hold in the action. Then

$$\begin{aligned} \delta_h(x_{(ij)} \cdot m_g) &= \delta_h(\lambda_g m_{(ij)g}) = \lambda_g \delta_h((ij)g) m_{(ij)g} = \lambda_g \delta_{(ij)h}(g) m_{(ij)g} \\ &= x_{(ij)} \cdot (\delta_{(ij)h} \cdot m_g) \end{aligned}$$

with $\lambda_g \in \Bbbk$ according to the definition of the action. Note that

$$x_{(ij)} \cdot (x_{(ik)} \cdot m_g) = \begin{cases} -a_{g^{-1}(ik)(ij)(ik)g}\, m_{(ij)(ik)g} & \text{if } \operatorname{sgn} g = -1, \\ -a_{g^{-1}(ik)g}\, m_{(ij)(ik)g} & \text{if } \operatorname{sgn} g = 1. \end{cases}$$

In any case, we have that $x_{(ij)}^2 \cdot m_g = -a_{g^{-1}(ij)g}\, m_g$ and

$$R_{(ij)(ik)} \cdot m_g = -\Big(\sum_{(st)\in\mathcal{O}_2^3} a_{g^{-1}(st)g}\Big) m_{(ij)(ik)g} = 0.$$

Let $W = \langle R_{(13)(23)}, R_{(23)(13)}, x_{(ij)}^2 : (ij) \in \mathcal{O}_2^3 \rangle$ and K be the subalgebra of $T(V_3)$ generated by W; K is a braided Hopf subalgebra because W is a Yetter-Drinfeld submodule contained in $\mathcal{P}(T(V_3))$ the primitive elements of $T(V_3)$. Then $K\#\Bbbk^{\mathbb{S}_3}$ is a Hopf subalgebra of $T(V_3)\#\Bbbk^{\mathbb{S}_3}$. For each $\mathbf{a} \in \mathfrak{A}_3$, by Lemma 28 we can define the algebra morphism $\psi = \psi_K \otimes \epsilon : K\#\Bbbk^{\mathbb{S}_3} \to \Bbbk$ where

$$\psi_{K|W[g]} = 0 \text{ if } g \neq e \text{ and } \psi_K(x_{(ij)}^2) = -a_{(ij)}\; \forall (ij) \in \mathcal{O}_2^3.$$

If J denotes the ideal of $K\#\Bbbk^{\mathbb{S}_3}$ generated by the generator of K, then $\psi^{-1} \rightharpoonup J \leftharpoonup \psi$ is the ideal generated by the generators of $\mathcal{I}_{\mathbf{a}}$. In fact, $\psi^{-1} =$

$\psi \circ \mathcal{S}$ is the inverse element of ψ in the convolution group $\mathrm{Alg}(K\#\Bbbk^{\mathbb{S}_3}, \Bbbk)$, $\mathcal{S}(W)[g] \subset (K\#\Bbbk^{\mathbb{S}_3})[g^{-1}]$ and $\mathcal{S}(x_{(ij)}^2) = -\sum_{h\in\mathbb{S}_3} \delta_{h^{-1}} x^2_{h^{-1}(ij)h}$. Then our claim follows if we apply $\psi \otimes \mathrm{id} \otimes \psi^{-1}$ to $(\Delta \otimes \mathrm{id})\Delta(x_{(ij)}^2) =$

$$= x^2_{(ij)} \otimes 1 \otimes 1 + \sum_{h\in\mathbb{S}_3} \delta_h \otimes x^2_{h^{-1}(ij)h} \otimes 1 + \sum_{h,g\in\mathbb{S}_3} \delta_h \otimes \delta_g \otimes x^2_{g^{-1}h^{-1}(ij)hg}$$

and $(\Delta \otimes \mathrm{id})\Delta(x) = x \otimes 1 \otimes 1 + x_{-1} \otimes x_0 \otimes 1 + x_{-2} \otimes x_{-1} \otimes x_0$ for $g \neq e$ and $x \in W[g]$; note that also $x_0 \in W[g]$.

The ideal $\psi^{-1} \rightharpoonup J$ is generated by

$$R_{(13)(23)}, \quad R_{(23)(13)} \quad \text{and} \quad x^2_{(ij)} + \sum_{g\in\mathbb{S}_3} a_{g^{-1}(ij)g}\delta_g \quad \forall (ij) \in \mathcal{O}_2^3.$$

Now $\mathcal{K}_{\mathbf{a}} = T(V_3)\#\Bbbk^{\mathbb{S}_3}/\langle \psi^{-1} \rightharpoonup J\rangle \neq 0$ because it has a non-zero quotient in $\mathrm{End}(M_3)$. Hence $\mathcal{A}_{[\mathbf{a}]}$ is monoidally Morita-Takeuchi equivalent to $\mathcal{B}(V_3)\#\Bbbk^{\mathbb{S}_3}$, by Theorem 4. □

5.2. **Hopf subalgebras and integrals of $\mathcal{A}_{[\mathbf{a}]}$.**

We collect some information about $\mathcal{A}_{[\mathbf{a}]}$. Let

$$\chi = \sum_{g\in\mathbb{S}_3} \mathrm{sgn}(g)\delta_g, \quad y = \sum_{(ij)\in\mathcal{O}_2^3} x_{(ij)}.$$

It is easy to see that χ is a group-like element and that $y \in \mathcal{P}_{1,\chi}(\mathcal{A}_{[\mathbf{a}]})$.

Proposition 30. *Let* $\mathbf{a} \in \mathfrak{A}_3$. *Then*

(a) $G(\mathcal{A}_{[\mathbf{a}]}) = \{1, \chi\}$.
(b) $\mathcal{P}_{1,\chi}(\mathcal{A}_{[\mathbf{a}]}) = \langle 1-\chi, y\rangle$.
(c) $\Bbbk\langle\chi, y\rangle$ *is isomorphic to the 4-dimensional Sweedler Hopf algebra.*
(d) *The Hopf subalgebras of* $\mathcal{A}_{[\mathbf{a}]}$ *are* $\Bbbk^{\mathbb{S}_3}$, $\Bbbk\langle\chi\rangle$ *and* $\Bbbk\langle\chi, y\rangle$.
(e) $\mathcal{S}^2(a) = \chi a \chi^{-1}$ *for all* $a \in \mathcal{A}_{[\mathbf{a}]}$.
(f) *The space of left integrals is* $\langle m_{top}\delta_e\rangle$; $\mathcal{A}_{[\mathbf{a}]}$ *is unimodular.*
(g) $(\mathcal{A}_{[\mathbf{a}]})^*$ *is unimodular.*
(h) $\mathcal{A}_{[\mathbf{a}]}$ *is not a quasitriangular Hopf algebra.*

Proof. We know that the coradical $(\mathcal{A}_{[\mathbf{a}]})_0$ of $\mathcal{A}_{[\mathbf{a}]}$ is isomorphic to $\Bbbk^{\mathbb{S}_3}$ by [AV]. Since $G(\mathcal{A}_{[\mathbf{a}]}) \subset (\mathcal{A}_{[\mathbf{a}]})_0$, (a) follows.

(b) Recall that $V_3 = M((12), \mathrm{sgn}) \in {}^{\Bbbk^{\mathbb{S}_3}}_{\Bbbk^{\mathbb{S}_3}}\mathcal{YD}$, see Subsection 2.1. Then $\mathcal{P}_{1,\chi}(\mathcal{A}_{[\mathbf{a}]})/\langle 1-\chi\rangle$ is isomorphic to the isotypic component of the comodule V_3 of type χ. That is, if $z = \sum_{(ij)\in\mathcal{O}_2^3} \lambda_{(ij)} x_{(ij)} \in (V_3)_\chi$, then

$$\delta(z) = \sum_{h\in G, (ij)\in\mathcal{O}_2^3} \mathrm{sgn}(h)\lambda_{(ij)}\delta_h \otimes x_{h^{-1}(ij)h} = \chi \otimes z.$$

Evaluating at $g \otimes \mathrm{id}$ for any $g \in \mathbb{S}_3$, we see that $\lambda_{(ij)} = \lambda_{(12)}$ for all $(ij) \in \mathcal{O}_2^n$. Then $z = \lambda_{(12)} y$. The proof of (c) is now evident.

(d) Let A be a Hopf subalgebra of $\mathcal{A}_{[\mathbf{a}]}$. Then $A_0 = A \cap (\mathcal{A}_{[\mathbf{a}]})_0 \subseteq \Bbbk^{\mathbb{S}_3}$ by [Mo, Lemma 5.2.12]. Hence A_0 is either $\Bbbk\langle\chi\rangle$ or else $\Bbbk^{\mathbb{S}_3}$. If $A_0 = \Bbbk\langle\chi\rangle$, then A is a pointed Hopf algebra with group $\mathbb{Z}/2$. Hence A is either $\Bbbk\langle\chi\rangle$ or else $\Bbbk\langle\chi, y\rangle$ by (b) and [N] or [CD][4]. If $A_0 = \Bbbk^{\mathbb{S}_3}$, then A is either $\Bbbk^{\mathbb{S}_3}$ or else $A = \mathcal{A}_{[\mathbf{a}]}$ by [AV].

To prove (e), just note that $\chi x_{(ij)}\chi^{-1} = -x_{(ij)}$.

(f) follows from Subsections 3.2 and 3.3. Let Λ be a non-zero left integral of $\mathcal{A}_{[\mathbf{a}]}$. By Lemma 8, the distinguished group-like element of $(\mathcal{A}_{[\mathbf{a}]})^*$ is ζ_h for some $h \in \mathbb{S}_3^{\mathbf{a}}$, hence $\Lambda\delta_h = \zeta_h(\delta_h)\Lambda = \Lambda$. Let us consider $\mathcal{A}_{[\mathbf{a}]}$ as a left $\Bbbk^{\mathbb{S}_3}$-module via the left adjoint action, see page 3. Let $\Lambda_g \in (\mathcal{A}_{[\mathbf{a}]})[g]$ such that $\Lambda = \sum_{g\in\mathbb{S}_3} \Lambda_g$. Then $\Lambda = \delta_e\Lambda = \sum_{s,t\in\mathbb{S}_3} \operatorname{ad}\delta_s(\Lambda_t)\delta_{s^{-1}}\delta_h = \Lambda_{h^{-1}}\delta_h$. Since $M_h \simeq \mathcal{A}_{[\mathbf{a}]}\delta_h$, we can use the lemmata of the Section 3 to compute Λ.

If $\mathbf{a}$ is generic, then $h = e$ by Theorem 1. Since $x_{(ij)}\Lambda = 0$ for all $(ij) \in \mathbb{S}_3$, $\Lambda = m_{\mathsf{top}}\delta_e$ by Lemma 10.

If $\mathbf{a}$ is sub-generic, we assume that $a_{(12)} \neq a_{(13)} = a_{(23)}$, then either $\Lambda = \Lambda_e\delta_e$ or $\Lambda_{(12)}\delta_{(12)}$ by Theorem 2. Since $x_{(ij)}\Lambda = 0$ for all $(ij) \in \mathbb{S}_3$, $\Lambda = m_{\mathsf{top}}\delta_e$ by Lemma 17 and Lemma 19.

(g) By (e), $\mathcal{S}^4 = \mathrm{id}$. By Radford's formula for the antipode and (f), the distinguished group-like element of $\mathcal{A}_{[\mathbf{a}]}$ is central, hence trivial. Therefore, $(\mathcal{A}_{[\mathbf{a}]})^*$ is unimodular.

(h) If there exists $R \in \mathcal{A}_{[\mathbf{a}]}\otimes\mathcal{A}_{[\mathbf{a}]}$ such that $(\mathcal{A}_{[\mathbf{a}]}, R)$ is a quasitriangular Hopf algebra, then $(\mathcal{A}_{[\mathbf{a}]}, R)$ has a unique minimal subquasitriangular Hopf algebra (A_R, R) by [R]. We shall show that such a Hopf subalgebra does not exist using (d) and therefore $\mathcal{A}_{[\mathbf{a}]}$ is not a quasitriangular Hopf algebra.

By [R, Prop. 2, Thm. 1] we know that there exist Hopf subalgebras H and B of $\mathcal{A}_{[\mathbf{a}]}$ such that $A_R = HB$ and an isomorphism of Hopf algebras $H^{*\mathrm{cop}} \to B$. Then $A_R \neq \mathcal{A}_{[\mathbf{a}]}$. In fact, let $M(d, \Bbbk)$ denote the matrix algebra over $\Bbbk$ of dimension d^2. Then the coradical of $(\mathcal{A}_{[\mathbf{a}]})^*$ is isomorphic to

- $\Bbbk^6$ if $\mathbf{a} = (0, 0, 0)$.
- $\Bbbk \oplus M(5, \Bbbk)^*$ if $\mathbf{a}$ is generic by Theorem 1.
- $\Bbbk^2 \oplus M(4, \Bbbk)^*$ if $\mathbf{a}$ is sub-generic by Theorem 2.

Since $(\mathcal{A}_{[\mathbf{a}]})_0 \simeq \Bbbk^{\mathbb{S}_3}$, $\mathcal{A}_{[\mathbf{a}]}$ is not isomorphic to $(\mathcal{A}_{[\mathbf{a}]})^{*\mathrm{cop}}$ for all $\mathbf{a} \in \mathfrak{A}_3$. Clearly, A_R cannot be $\Bbbk^{\mathbb{S}_3}$. Since $\mathcal{A}_{[\mathbf{a}]}$ is not cocommutative, R cannot be $1\otimes 1$. The quasitriangular structures on $\Bbbk\langle\chi\rangle$ and $\Bbbk\langle\chi, y\rangle$ are well known, see for example [R]. Then it remains the case $A_R \subseteq \Bbbk\langle\chi, y\rangle$ with $R = R_0 + R_\alpha$ where $R_0 = \frac{1}{2}(1\otimes 1 + 1\otimes\chi + \chi\otimes 1 - \chi\otimes\chi)$ and $R_\alpha = \frac{\alpha}{2}(y\otimes y + y\otimes\chi y + \chi y\otimes\chi y - \chi y\otimes y)$ for some $\alpha \in \Bbbk$. Since $\Delta(\delta_g)^{\mathrm{cop}}R = R\Delta(\delta_g)$ for all $g \in \mathbb{S}_3$, then

$$\Delta(\delta_g)^{\mathrm{cop}}R_0 = R_0\Delta(\delta_g) = \Delta(\delta_g)R_0 \quad \text{in } \Bbbk^{\mathbb{S}_3};$$

but this is not possible because $R_0^2 = 1\otimes 1$ and $\Bbbk^{\mathbb{S}_3}$ is not cocommutative. $\square$

[4]The classification of all finite dimensional pointed Hopf algebras with group $\mathbb{Z}/2$ also follows easily performing the Lifting method [AS].

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